ABUNDANCE OF HETEROCLINIC AND HOMOCLINIC ORBITS FOR THE HYPERCHAOTIC RÖSSLER SYSTEM.

DANIEL WILCZAK

DEPARTAMENT OF MATHEMATICS, UNIVERSITY OF BERGEN,
JOHANNES BRUNSGATE 12, 5008 BERGEN, NORWAY

Abstract. The four dimensional Rössler system is investigated. For this system the Poincaré map exhibits chaotic dynamics with two expanding directions and one strongly contracting direction. It is shown that the 16th iterate of this Poincaré map has a nontrivial invariant set on which it is semiconjugated to the full shift on two symbols. Moreover, it is proven that there exist infinitely many homoclinic and heteroclinic solutions connecting periodic orbits of period two and four, respectively. The proof utilizes the method of covering relations with smooth tools (cone conditions).

The proof is computer assisted - interval arithmetic is used to obtain bounds of the Poincaré map and its derivative.

1. Introduction

In recent years several computer assisted methods for proving the existence of chaotic dynamics for maps and flows have been developed. They are often based on some topological tools like the Conley index [MM, P00, P03], the Brouwer degree [ZGi, Z] or smooth ones like the shadowing theorem [SK]. In these approaches chaos is understood as a symbolic dynamics embedded in the full dynamics. In [T] a careful study of a normal form field around the equilibrium combined with global analysis of the Poincaré map and the cone field have been used to prove that the Lorenz system admits a strange attractor witch is also a support of the unique Sinai-Ruelle-Bowen measure for the flow.

The study of the existence of homoclinic or heteroclinic orbits is a much harder task than verifying the existence of symbolic dynamics, since it requires some tools guaranteeing convergence. Here we can use some specific properties of the system

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like the existence of a local Lyapunov function around equilibria or a gradient structure of the system. In the general case, some information about the derivatives of the system is helpful to solve problem of convergence [AM, AZ, GZ, SK, WZ] even in the nonhyperbolic situation [KWZ]. For proving the existence of homoclinic or heteroclinic orbits between periodic orbits of an ODE these methods require solving of variational equations for this ODE.

The problem becomes much harder when the dimension of the phase space increases, especially if the number of observed unstable directions is larger. Up to now, the above mentioned methods of proving the existence of homoclinic and heteroclinic orbits have been successfully adopted to the systems with a onedimensional unstable direction.

Some of the well known dynamical systems which exhibits complicated dynamics are the three dimensional Rössler system [R76] and the four dimensional Rössler system [R79]. The first is given by a system of ordinary differential equations

$$\left\{\begin{array}{lcl} \dot{x} &=& -y-z \\ \dot{y} &=& x+by \\ \dot{z} &=& b+z(x-a). \end{array}\right. \tag{1}$$

Zgliczyński [Z] proved that for the original parameter values given by Rössler, i.e. $a = 5.7$, $b = 0.2$ a suitable Poincaré map is semiconjugated to the symbolic dynamics with a nontrivial invariant set. Later, Pilarczyk [P00, P03] gave a computer assisted proof that for parameter values $a = 2.2$ and $a = 3.1$ a periodic orbits exist. Very recently, using the method based on the Lyapunov-Schmidt reduction and the rigorous integration of the third order variational equations it was proven that in the range of parameter values $a \in [2.83244, 3.837358168411]$ there exist two period doubling bifurcations connected by a branch of period two (with respect to the Poincaré map) orbits.

The four dimensional Rössler system [R79] is an ODE given by

$$\begin{cases} \dot{x} = -y - w \\ \dot{y} = x + ay + z \\ \dot{z} = dz + cw \\ \dot{w} = xw + b \end{cases} \tag{2}$$

A Poincaré map of this system exhibits a strongly attracting invariant set with two expanding directions and one contracting direction. In [Li] a strong numerical evidence of the existence of symbolic dynamics on two symbols for the ninth iterate of this Poincaré map is presented. Li in his paper [Li] notes that a rigorous verification of assumptions of the main topological theorem by means of interval analysis requires terrible long time and because of that only a nonrigorous extrapolation method is used to argue the conjecture.

In this paper we present a computer assisted proof of the existence of chaotic dynamics and abundance of homoclinic and heteroclinic orbits between two periodic orbits in the four dimensional Rössler system. The method utilizes topological tools of the covering relations [ZGi] with the cone conditions.

The paper is organized as follows. In Section 2 we introduce the method of covering relations [ZGi] to make this paper more self-consistent. In Section 3 we give the notion of cone conditions and we prove the main theorem which can be used as a tool for proving the existence of homoclinic and heteroclinic solutions. In Section 4 we present the main theorem about the Rössler system.

The existence of chaotic dynamics and abundance of homoclinic and heteroclinic orbits is proved by means of interval analysis. The proof requires the computation of rigorous bounds for a Poincaré map together with its derivatives. The C++ program which realizes the necessary computations is available at [W].

2. Topological tools

In this section we present main topological tools used in this paper. The crucial notion is that of *covering relation* [ZGi].

2.1. h-sets.

Notation: For a given norm in $\mathbb{R}^n$ by $B_n(c,r)$ we will denote an open ball of radius r centered at $c \in \mathbb{R}^n$. When the dimension n is obvious from the context we will drop the subscript n. Let $S^n(c,r) = \partial B_{n+1}(c,r)$, by the symbol S^n we will denote $S^n(0,1)$. We set $\mathbb{R}^0 = \{0\}$, $B_0(0,r) = \{0\}$, $\partial B_0(0,r) = \emptyset$.

For a given set Z, by $\operatorname{int} Z$, $\overline{Z}$, ∂Z we denote the interior, the closure and the boundary of Z, respectively. For the map $h : [0,1] \times Z \to \mathbb{R}^n$ we set $h_t = h(t,\cdot)$. By Id we denote the identity map. By $f : X \multimap Y$ we denote a partial map, i.e. a map which domain is not necessary whole X. For a map $f : X \multimap Y$, by $\operatorname{dom}(f)$ we will denote the domain of f. For $N \subset \Omega$, N-open and $c \in \mathbb{R}^n$ by $\deg(f, N, c)$ we denote the local Brouwer degree. For the properties of this notion we refer the reader to [L] (see also Appendix in [ZGi]).

Definition 1. [ZGi, Definition 1] *An h-set, N, is the object consisting of the following data*

- $|N|$ *- a compact subset of* $\mathbb{R}^n$
- $u(N), s(N) \in \{0, 1, 2, \dots\}$, *such that* $u(N) + s(N) = n$
- *a homeomorphism* $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$, *such that*

$$c_N(|N|) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1).$$

We set

$$N_c = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1),$$
$$N_c^- = \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1)$$
$$N_c^+ = \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1)$$
$$N^- = c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+)$$

Hence an h-set, N, is a product of two closed balls in some coordinate system. The numbers, $u(N)$ and $s(N)$, stand for the dimensions of nominally unstable and stable directions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Observe that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$. In the sequel to make notation less cumbersome we will often drop the bars in the symbol $|N|$ and we will use N to denote both the h-sets and its support.

2.2. **Covering relations.**

Definition 2. [ZGi, Definition 6] *Assume that N, M are h-sets, such that $u(N) = u(M) = u > 0$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. Let w be a nonzero integer. We say that*

$$N \overset{f,w}{\Longrightarrow} M$$

(N f-covers M with degree w) iff the following conditions are satisfied

1: *there exists a continuous homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true*

$$h_0 = f_c, \tag{3}$$
$$h([0,1], N_c^-) \cap M_c = \emptyset, \tag{4}$$
$$h([0,1], N_c) \cap M_c^+ = \emptyset. \tag{5}$$

2: *There exists a map* $A : \mathbb{R}^u \to \mathbb{R}^u$*, such that*

$$h_1(p,q) = (A(p),0), \text{ for } p \in \overline{B_u}(0,1) \text{ and } q \in \overline{B_s}(0,1), \tag{6}$$

$$A(\partial B_u(0,1)) \subset \mathbb{R}^u \setminus \overline{B_u}(0,1). \tag{7}$$

Moreover, we require that

$$\deg(A, \overline{B_u}(0,1), 0) = w,$$

Intuitively, $N \overset{f}{\Longrightarrow} M$ if f stretches N in the 'nominally unstable' direction, so that its projection onto 'unstable' direction in M covers in topologically nontrivial manner projection of M. In the 'nominally stable' direction N is contracted by f. As a result N is mapped across M in the unstable direction, without touching M^+. It is also very helpful to note that the degree w in the covering relation depends only on $A_{|\partial B_u(0,1)}$.

The geometry of Definition 2 is presented in Fig. 1.

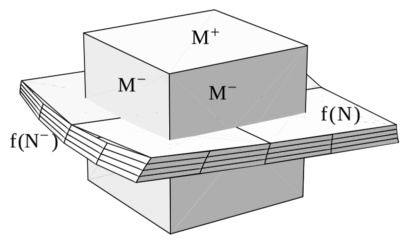

FIGURE 1. An illustration of the covering relation $N \overset{f}{\Longrightarrow} M$. In this case $u(N) = u(M) = 2$ and $s(N) = s(M) = 1$.

Definition 3. *Assume* $(N_i)_{i\in\mathbb{Z}}$ *are h-sets in* $\mathbb{R}^n$ *and* $f_i : |N_i| \to \mathbb{R}^n$ *are continuous. We say that a sequence* $(x_i)_{i\in\mathbb{Z}}$ *is a* full orbit with respect to $(f_i)_{i\in\mathbb{Z}}$ and $(N_i)_{i\in\mathbb{Z}}$ *iff*

$$\begin{aligned} x_i &\in \operatorname{int}|N_i|, \quad i \in \mathbb{Z} \\ f_i(x_i) &= x_{i+1}, \quad i \in \mathbb{Z} \end{aligned}$$

If additionally there exists $T > 0$ such that

$$\begin{aligned} N_{i+T} &= N_i, \quad i \in \mathbb{Z} \\ f_{i+T} &= f_i, \quad i \in \mathbb{Z} \\ x_{i+T} &= x_i, \quad i \in \mathbb{Z} \end{aligned}$$

then we say that $(x_0, \ldots, x_{T-1})$ is a periodic orbit *with respect to $(f_0, \ldots, f_{T-1})$ and $(N_0, N_1, \ldots, N_{T-1})$.*

The following theorem was proved in [ZGi].

Theorem 4. [ZGi, Theorem 9] *Assume $(N_i)_{i\in\mathbb{Z}}$ are h-sets and for each $i \in \mathbb{Z}$ we have*

$$N_i \overset{f_i,w_i}{\Longrightarrow} N_{i+1}$$

Then there exists a full orbit $(x_i)_{i\in\mathbb{Z}}$ with respect to $(f_i)_{i\in\mathbb{Z}}$ and $(N_i)_{i\in\mathbb{Z}}$. Moreover, if the sequence of sets and functions is periodic, i.e. there exists $T > 0$ such that $N_{i+T} = N_T$, $f_{i+T} = f_i$ for $i \in \mathbb{Z}$ then this full orbit $(x_i)_{i\in\mathbb{Z}}$ may be chosen such that $(x_0, \ldots, x_{T-1})$ is a periodic orbit with respect to $(f_0, \ldots, f_{T-1})$ and $(N_0, N_1, \ldots, N_{T-1})$.

Obviously we cannot make any claim about the uniqueness of $(x_i)_{i\in\mathbb{Z}}$ in Theorem 4.

3. Cone conditions

The goal of this section is to introduce a method for proving the uniqueness of trajectory in a periodic sequence of covering relations. Moreover, this method give us a tool for proving convergence of a trajectory to a fixed point or periodic orbit.

Definition 5. [KWZ, Definition 10] *Let $N \subset \mathbb{R}^n$ be an h-set and $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form*

$$Q(x) = Q(p, q) = \alpha(p) - \beta(q), \qquad x = (p, q) \in \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)},$$

where $\alpha : \mathbb{R}^{u(N)} \to \mathbb{R}$, and $\beta : \mathbb{R}^{s(N)} \to \mathbb{R}$ are positive definite quadratic forms.

The pair (N, Q) will be called an h-set with cones.

Quite often we will drop Q in the symbol (N, Q) and we will say that N is an h-set with cones.

Definition 6. [KWZ, Definition 13] *Assume that $(N, Q_N), (M, Q_M)$ are h-sets with cones, such that $u(N) = u(M) = u$ and let $f : N \to \mathbb{R}^n$ be continuous. Assume that $N \overset{f}{\Longrightarrow} M$. We say that f satisfies the cone condition (with respect to the pair (N, M)), if any $x_1, x_2 \in N_c$ with $x_1 \neq x_2$ satisfy*

$$Q_M(f_c(x_1) - f_c(x_2)) > Q_N(x_1 - x_2).$$

Whenever it is convenient, we will also say that the cone conditions are satisfied for the covering relation $N \overset{f}{\Longrightarrow} M$, if the above condition is satisfied.

The idea of this definition is presented in Figure 2. In particular, the image of the positive cone $Q_N^+(x_2) = \{x \in N_c : Q_N(x - x_2) > 0\}$ must be mapped by f_c into the positive cone with respect to Q_M at $f_c(x_2)$.

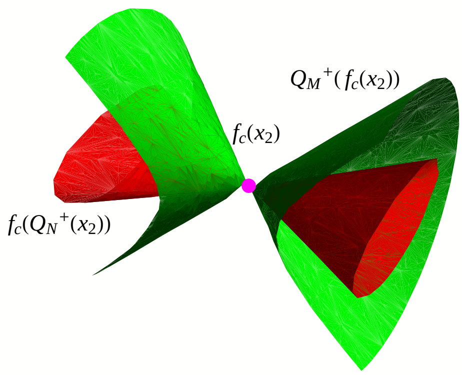

FIGURE 2. Geometry of the cone conditions. At any point $x_2 \in |N|$, the positive cone $Q_N^+(x_2)$ must be mapped into the positive cone $Q_M^+(f(x_2))$.

We would like to use the cone conditions as a tool for proving the existence of homoclinic and heteroclinic orbits. We start with the following

Lemma 7. *Assume that*

$$N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} N_2 \overset{f_2}{\Longrightarrow} \cdots \overset{f_{T-1}}{\Longrightarrow} N_T = N_0,$$

where (N_i, Q_i) are h-sets with cones and f_i satisfies the cone condition with respect to the pair (N_i, N_{i+1}), $i = 0, \ldots, T-1$. Then there exists a unique periodic orbit $(x_0, \ldots, x_{T-1})$ with respect to $(f_0, \ldots, f_{T-1})$ and $(N_0, N_1, \ldots, N_{T-1})$.

Proof: Without loss of generality we may assume that the homeomorphisms c_{N_i} from Definition 1 are identities, so we have $|N_i| = (N_i)_c$ and $f_i = (f_i)_c$ for $i = 0, \dots, n-1$. The existence of a periodic orbit with respect to $(f_0, \dots, f_{T-1})$ and $(N_0, N_1, \dots, N_{T-1})$ follows from Theorem 4. Assume $(x_0, \dots, x_{T-1})$, $(y_0, \dots, y_{T-1})$ are two different periodic orbits with respect to $(f_0, \dots, f_{T-1})$ and $(N_0, N_1, \dots, N_{T-1})$. By Definition 3 this means that $x_i, y_i \in |N_i|$ for $i = 0, \dots, T-1$. Without loss of generality we may assume that $x_0 \neq y_0$. Since each f_i satisfies the cone conditions with respect to the pair (N_i, N_{i+1}) we have

$$
\begin{aligned}
Q_0(x_0 - y_0) = Q_0\left(f_{T-1}(x_{T-1}) - f_{T-1}(y_{T-1})\right) \geq \\
\geq Q_{T-1}\left(f_{T-2}(x_{T-2}) - f_{T-2}(y_{T-2})\right) \geq \cdots \geq Q_1\left(f_0(x_0) - f_0(y_0)\right) > \\
> Q_0(x_0 - y_0)
\end{aligned}
$$

which is a contradiction. ∎

The following theorem gives us a tool for proving the existence of homoclinic and heteroclinic orbits for maps.

Theorem 8. *Assume that*

$$
\begin{aligned}
N_0 \stackrel{f_0}{\Longrightarrow} N_1 \stackrel{f_1}{\Longrightarrow} N_2 \stackrel{f_2}{\Longrightarrow} \cdots \stackrel{f_{n-1}}{\Longrightarrow} N_n = N_0, \\
N_0 \stackrel{h_0}{\Longrightarrow} M_1 \stackrel{h_1}{\Longrightarrow} M_2 \stackrel{h_2}{\Longrightarrow} \cdots \stackrel{h_{m-2}}{\Longrightarrow} M_{m-1} \stackrel{h_{m-1}}{\Longrightarrow} K_0 \\
K_0 \stackrel{g_0}{\Longrightarrow} K_1 \stackrel{g_1}{\Longrightarrow} K_2 \stackrel{g_2}{\Longrightarrow} \cdots \stackrel{g_{k-1}}{\Longrightarrow} K_k = K_0,
\end{aligned}
$$

where

- (N_i, Q_{N_i}) *are h-sets with cones and* f_i *satisfies the cone condition with respect to* (N_i, N_{i+1}), $i = 0, \dots, n-1$
- (K_i, Q_{K_i}) *are h-sets with cones and* g_i *satisfies the cone condition with respect to* (K_i, K_{i+1}), $i = 0, \dots, k-1$

Then there exist

1. *a unique periodic orbit* $(u_0, \dots, u_{n-1})$ *with respect to* $(f_0, \dots, f_{n-1})$ *and* $(N_0, N_1, \dots, N_{n-1})$
2. *a unique periodic orbit* $(v_0, \dots, v_{k-1})$ *with respect to* $(g_0, \dots, g_{k-1})$ *and* $(K_0, K_1, \dots, N_{k-1})$

(3) *a full orbit* $(x_i)_{i\in\mathbb{Z}}$ *with respect to*

$$(\dots, f_0, \dots, f_{n-1}, f_0, \dots, f_{n-1},\\ h_0, \dots, h_{m-1},\\ g_0, \dots, g_{k-1}, g_0, \dots, g_{k-1}, \dots) \tag{8}$$

and the sequence

$$\big((N_1, N_2, \dots, N_n)^{\mathbb{N}}, M_1, M_2, \dots, M_{m-1}, (K_0, K_1, \dots, K_{k-1})^{\mathbb{N}}\big) \tag{9}$$

such that

$$x_0 \in |N_0|$$

$$\lim_{t\to\infty} x_{tk+m} = v_0 \tag{10}$$

$$\lim_{t\to-\infty} x_{tn} = u_0 \tag{11}$$

Proof: The existence of a full orbit $(x_i)_{i\in\mathbb{Z}}$ with respect to (8) and (9) follows from Theorem 4. Assertions 1 and 2 are direct consequences of Lemma 7. It remains for us to show convergence properties (10-11).

Without loss of generality we may assume that homeomorphisms c_{K_i} and c_{N_j} from Definition 1 are identities, so we have $|K_i| = (K_i)_c$, $|N_j| = (N_j)_c$ and $g_i = (g_i)_c$, $f_j = (f_j)_c$ for $i = 0, \dots, k-1$, $j = 0, \dots, n-1$.

First we will prove (10). Put $g = g_{k-1} \circ \dots \circ g_0$. Observe that for $z \in \mathrm{dom}(g) \subset |K_0|$ such that $g(z) \in |K_0|$ the function

$$V_K(z) = Q_{K_0}(z - v_0)$$

is strictly increasing along trajectory of g, i.e.

$$V_K(z) < V_K(g(z))$$

Since $g^t(x_m) = x_{tk+m} \in |K_0|$ for $t > 0$ using a standard argument of Lyapunov functions we obtain that the ω-limit set of x_m is a single point and it must be equal to v_0, hence (10) holds true.

In a similar way we can prove (11). Put $f = f_{n-1} \circ \dots \circ f_0$. Then

$$V_N(z) = Q_{N_0}(z - u_0)$$

is strictly increasing along trajectories of f. This proves that the α-limit set of x_0 is a single point and it must be equal to u_0. ∎

It should be noted that the verification that a smooth function satisfies the cone conditions with respect to a pair of h-sets can be reduced to the verification that some matrix is positive definite in the following way.

Let $(N, Q_N), (M, Q_M)$ be h-sets with cones and let $N \overset{f}{\Longrightarrow} M$. Assume $f_c : N_c \to \mathbb{R}^n$ is of class $\mathcal{C}^1$. Let $[Df_c(N_c)]_I$ be an interval enclosure of Df_c on N_c, i.e. $[Df_c(N_c)]_I$ is a convex subset of $\mathbb{R}^{n\times n}$ such that $Df_c(x) \in [Df_c(N_c)]_I$ for $x \in N_c$. Let us identify the quadratic form Q_M with a symmetric matrix which defines this form.

Lemma 9. [KWZ, §5.3] *If the interval matrix*

$$V = [Df_c(N_c)]_I^T Q_M [Df_c(N_c)]_I - Q_N$$

is positive definite then f satisfies the cone conditions with respect to (N, M).

Proof. Let us fix any $x_1, x_2 \in N_c$. We have to prove that $Q_M(f_c(x_2) - f_c(x_1)) - Q_N(x_2 - x_1) > 0$. We have

$$f_c(x_2) - f_c(x_1) = \int_0^1 Df_c(x_1 + t(x_2 - x_1))(x_2 - x_1)dt = B(x_2 - x_1),$$

where

$$B = \int_0^1 Df_c(x_1 + t(x_2 - x_1))dt \in [Df_c(N_c)]_I.$$

From the above we have

$$\begin{aligned} Q_M(f(x_2) - f(x_1)) - Q_N(x_2 - x_1) &= Q_M(B(x_2 - x_1)) - Q_N(x_2 - x_1) \\ &= (x_2 - x_1)^T B^T Q_M B(x_2 - x_1) - (x_2 - x_1)^T Q_N(x_2 - x_1) \\ &= (x_2 - x_1)^T \left(B^T Q_M B - Q_N\right)(x_2 - x_1) > 0 \end{aligned}$$

since $B^T Q_M B - Q_N \in V$ is positive definite matrix. ∎

4. Application to the Rössler system.

In [R79] Rössler observed that for that for parameters $a = 0.25$, $b = 3$, $c = -0.5$ and $d = 0.05$ the Poincaré map of the system (2) exhibits chaotic dynamics with an attractor possessing two expanding directions. Such type of systems are often called hyperchaotic. A projection of a typical trajectory of this system close to the attractor is presented in Figure 3 - see also [Li, Figure 2].

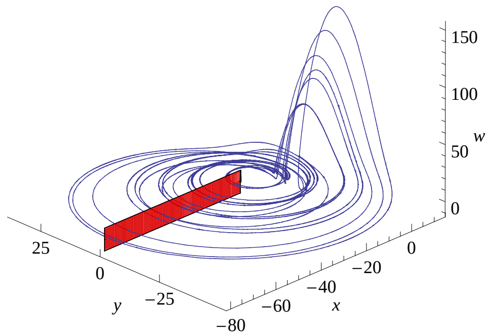

FIGURE 3. Typical trajectory of the system (2) and the Poincaré section – see also [Li, Figure 2].

Let us choose a Poincaré section

$$\Pi = \{(x, y, z, w) \in \mathbb{R}^4 : y = 0, \dot{y} = x + z < 0\}$$

and denote by $P : \Pi \nrightarrow \Pi$ the Poincaré map (see also Figure 3). Using the standard Newton method we find two approximate fixed points for P^2 and P^4, respectively. Let us denote them by

$$\begin{aligned} x_A &= (-37.6927329080591, 21.3668411884911, 0.079619614197614) \\ x_B &= (-46.621314119875045, 26.63066048091909, 0.06436282304302137) \quad (12) \\ x_{B'} &= P^2(x_B) \end{aligned}$$

In Figure 4 these periodic orbits projected onto the (x, y, w) coordinates are presented.

Let us use the notatin $\Sigma_k = \{1, 2, \ldots, k\}^{\mathbb{Z}}$. On Σ_k we define the shift map σ by

$$\sigma\left((a_i)_{i\in\mathbb{Z}}\right) = (a_{i+1})_{i\in\mathbb{Z}}$$

We will recall the well known definition which is often used to describe chaotic systems, see for example [D].

Definition 10. *We say that a map $f : X \to X$ is Σ_k-chaotic if there exists a continuous surjection $\pi : X \to \Sigma_k$ such that $\sigma \circ \pi = \pi \circ f$ and if the preimage of any periodic sequence from Σ_k contains a periodic point of f with the same principal period.*

Now we can state the main theorem of this section.

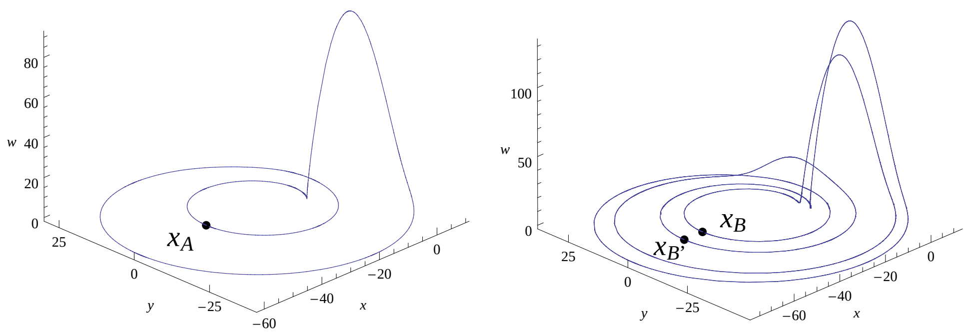

FIGURE 4. Periodic orbits (left panel) x_A and (right panel) x_B.

Theorem 11. *There exists a subset* $Q \subset \Pi$ *such that the map* $P^{16}|_Q : Q \to Q$ *is well defined and* Σ_2*-chaotic. Moreover, there exist two periodic points for* P *of period* 2 *and* 4 *respectively, and there exist*

- *infinitely many heteroclinic orbits between them in both directions*
- *infinitely many homoclinic orbits to both of them*

Remark 12. *In* [Li]*, strong numerical evidence of the chaotic dynamics is given just for the ninth iterate of the Poincaré map. The reason for us using a much higher iterate of the Poincaré map in Theorem 11 is that it is very difficult to verify the cone conditions on a significantly larger neighborhood around both periodic points. This fact is due to the weak hyperbolicity in one direction at these points. Therefore, to be able to apply our method of cone conditions, we decrease the size of the neighborhoods. This allows us to rigorously verify the cone conditions in an acceptable CPU time. As a consequence, a point from a small vicinity of one periodic orbit needs more iterations to reach the small vicinity of the second periodic point.*

It seems, the symbolic dynamics for the lower iterate of the Poincaré map requires use of very large sets as it is shown in [Li].

In order to prove Theorem 11 we use the methods described in Theorem 4 and Theorem 8. First, we construct h-sets with cones around periodic points and verify the existence of covering relations with cones on them. Next we construct sequences of covering relations between these periodic chains of coverings in both directions. Before we give details we introduce some notations.

4.1. **Representation of h-sets.** For a vector $r \in \mathbb{R}^n$ by $\mathrm{Diag}(r)$ we will denote a diagonal matrix M with coefficients $M_{ii} = r_i$, $i = 1, \ldots, n$. In the proof of Theorem 11 we will use three dimensional h-sets with two nominally unstable directions and one nominally stable direction only. Additionally, our h-sets will be always parallelepipeds. Therefore we will use the following notation to define an h-set. Let $x, r = (r_1, r_2, r_3) \in \mathbb{R}^3$ and $L \in \mathbb{R}^{3\times 3}$ be such that $r_1, r_2, r_3 > 0$ and let $L\mathrm{Diag}(r)$ an isomorphism.

An h-set $(N, c_N) = h(x, L, r)$ is defined in the following way:

$$\begin{aligned} |N| &= \{x + L\mathrm{Diag}(r)u : \|u\| \leq 1)\} \\ u(N) &= 2,\ s(N) = 1 \\ c_N(u) &= (L\mathrm{Diag}(r))^{-1}(u - x) \end{aligned}$$

In this representation L will usually be a matrix with normalized columns so the vector r gives us information about the size of the h-set in the directions given by the matrix L. The vector x is obviously the center of parallelepiped.

4.2. **Periodic points and cone conditions.** Let x_A, x_B and $x_{B'}$ be as in (12). The point x_A is an approximate period two point for the Poincaré map P with eigenvalues of $DP^2(x_A)$ close to 5.14, 1.84 and $1.25\cdot 10^{-17}$ and x_B is an approximate period four point with eigenvalues of $DP^4(x_B)$ close to -79.58, 1.94, $-6.5\cdot 10^{-19}$. Columns of the following matrix

$$L_A = \begin{bmatrix} 0.029506713712874 & 0.82097852244306 & 0.026504078953280 \\ 0.99956458027526 & -0.57095643711962 & 0.013179779535486 \\ 0.000060868770038109 & 0.0017356839801992 & 0.99956181760321 \end{bmatrix}$$

are approximate normalized (with respect to Euclidean norm) eigenvectors of the derivative of P^2 at x_A. Put

$$\begin{aligned} r_A &= 10^{-5}(10, 100, 1) \\ A &= h(x_A, L_A, 1.4 r_A) \end{aligned}$$

In a similar way we will define h-sets around x_B and $x_{B'} = P^2(x_B)$. Matrices

$$L_B = \begin{bmatrix} -0.98674987281146 & -0.84986692848658 & 0.021434994608289 \\ -0.16224312694060 & 0.52699603973067 & 0.010667663623670 \\ -0.0013624489376921 & -0.0011738709323310 & 0.99971332988960 \end{bmatrix}$$

$$L_{B'} = \begin{bmatrix} -0.67010430754231 & -0.89431829035560 & 0.018308827945424 \\ -0.74226663806886 & 0.44743042386518 & 0.0091238097725906 \\ -0.00067455407009 & -0.0009007415045141 & 0.9997907495643 \end{bmatrix}$$

are built of the normalized eigenvectors of $DP^4(x_B)$ and $DP^4(x_{B'})$, respectively. Put

$$\begin{aligned} r_B &= 10^{-6}(10, 100, 1) \\ r_{B'} &= 10^{-6}(6, 100, 1) \\ B &= h(x_B, L_B, r_B) \\ B' &= h(x_{B'}, L_{B'}, r_{B'}) \end{aligned}$$

For h-sets A, B and B' we set the quadratic form which defines cones to be the same and given by the following diagonal matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{13}$$

The following lemma is the first step in the proof of Theorem 11.

Lemma 13. *The following covering relations hold*

$$A \overset{P^2}{\Longrightarrow} A$$

$$B \overset{P^2}{\Longrightarrow} B' \overset{P^2}{\Longrightarrow} B$$

Moreover, the cone conditions for these covering relations are satisfied.

Proof: The existence of covering relations has been proven using a rigorous ODE solver [CAPD] and the algorithms presented in [WZ2, §6]. Under the assumption that the Poincaré map exists on an h-set it is enough to verify some inequalities for the Poincaré map on the boundary of this h-set - as it is proven in [WZ2, Lemmas 9,10]. We apply those algorithms with grids of the boundary and settings of the Taylor method listed in Table 1.

In order to verify the existence of the Poincaré map and the cone conditions we apply the $\mathcal{C}^1$-Lohner algorithm [Z1] implemented in [CAPD] to solve the first order variational equations for the system (2) and to compute rigorous bounds for the

h-set	pair of walls in coordinates of h-set	order	step	grid
A	$\{-1,1\} \times [-1,1] \times [-1,1]$	6	0.01	$1 \times 55 \times 1$
A	$[-1,1] \times \{-1,1\} \times [-1,1]$	6	0.01	$20 \times 1 \times 1$
A	$[-1,1] \times [-1,1] \times \{-1,1\}$	7	0.008	$6 \times 30 \times 1$
B	$\{-1,1\} \times [-1,1] \times [-1,1]$	5	0.002	$1 \times 20 \times 2$
B	$[-1,1] \times \{-1,1\} \times [-1,1]$	5	0.002	$20 \times 1 \times 1$
B	$[-1,1] \times [-1,1] \times \{-1,1\}$	7	0.005	$2 \times 20 \times 1$
B'	$\{-1,1\} \times [-1,1] \times [-1,1]$	5	0.002	$1 \times 40 \times 1$
B'	$[-1,1] \times \{-1,1\} \times [-1,1]$	5	0.002	$100 \times 1 \times 6$
B'	$[-1,1] \times [-1,1] \times \{-1,1\}$	7	0.005	$2 \times 10 \times 1$

TABLE 1. Order and time step of the Taylor method and grids for the algorithms verifying the existence of covering relations $A \Longrightarrow A$, $B \Longrightarrow B' \Longrightarrow B$.

h-set	order of Taylor method	time step	grid (in coordinates of h-set)
A	6	0.0035	$40 \times 30 \times 1$
B	6	0.0035	$14 \times 14 \times 1$
B'	5	0.003	$110 \times 75 \times 1$

TABLE 2. Parameters of the $\mathcal{C}^1$-Lohner algorithm when compute derivatives of Poincaré map.

derivatives of the Poincaré map. The settings and number of elements into which we divided each h-set are listed in Table 2. As a result we obtain the following estimations

$$DP^2(|A|) \subset \begin{bmatrix} 1.92 \pm 0.0205 & 0.0951 \pm 0.0302 & -0.0521 \pm 0.000928 \\ 2.24 \pm 0.0162 & 5.08 \pm 0.0238 & -0.126 \pm 0.000735 \\ 0.00404 \pm 0.00739 & 0.000193 \pm 0.0114 & -0.00011 \pm 0.000347 \end{bmatrix}$$

$$DP^2(|B|) \subset \begin{bmatrix} 2.64 \pm 0.0335 & 1.22 \pm 0.0515 & -0.0696 \pm 0.00125 \\ 2.29 \pm 0.0141 & 5.21 \pm 0.0217 & -0.105 \pm 0.000529 \\ 0.00266 \pm 0.0038 & 0.00122 \pm 0.00598 & -0.00007 \pm 0.000145 \end{bmatrix}$$

$$DP^2(|B'|) \subset \begin{bmatrix} -8.01 \pm 0.00344 & -18.1 \pm 0.00681 & 0.312 \pm 0.000127 \\ -1.87 \pm 0.00193 & -2.47 \pm 0.00383 & 0.0568 \pm 0.0000708 \\ -0.0111 \pm 0.00204 & -0.0249 \pm 0.00404 & 0.00043 \pm 0.0000742 \end{bmatrix}$$

One can check that these matrices expressed in coordinate systems of h-sets A, B and B', as required in the Definition 6 of the cone conditions, satisfy the assumption of Lemma 9 with Q as defined in (13). This proves that the cone conditions are satisfied for covering relations asserted in this lemma.

∎

4.3. **Heteroclinic connections between periodic orbits.** The second step of the proof of Theorem 11 is to construct two chains of covering relations from a vicinity of A to B and from a vicinity of B to A. This together with Theorem 4 will give us the existence of chaotic dynamics for a suitable iterate of the Poincaré map. Since the sets A and B are very small we first enlarge them by a sequence of covering relations and then we will obtain the chaotic dynamics between the largest sets in the neighborhood of A and B, respectively.

Define

$$M_i = h\left(x_A, L_A, 1.4 \cdot (1.675)^{i+1}\, r_A\right)$$

for $i = 0, \ldots, 7$. Put

$$x_8 = (-37.812397400826434, 21.449972776479257, 0.079367425708741862)$$

The point x_8 is chosen as an approximate heteroclinic point between x_A and x_B – see Figure 5. This point will be the center of the first set in the heteroclinic chain. The next points are chosen on the trajectory (or close to the trajectory) of x_8. To be more precise, put $x_{i+1} = P^2(x_i)$ for $i = 8, 9, \ldots, 13$.

It should be noted that we do not need to compute the exact trajectory of x_8. The actual coordinates of the points x_i (computed only approximately by a nonrigorous routine) are written to the output file `centersOfSets.dat` by the C++ program realizing the computer assisted proof. Put

$$M_i \quad = \quad h\left(x_i, L_A, (1.36)^i r_A\right)$$

for $i = 8, \ldots, 14$. All h-sets in this sequence use the same coordinate system as the h-set A. The reason is that most of these points are very close to the point A. Only at the last point could we choose significantly better coordinate system. Since the computational part is easy here it is not necessary to find better expanding and contracting directions.

Finding of heteroclinic connection from B to A is more difficult, since the heteroclinic orbit escapes from a vicinity of B along the very weak expanding direction. Moreover, it is very difficult to verify the existence of the cone conditions for $B' \overset{P^2}{\Longrightarrow} B$ on significantly larger sets than B and B'. Since the size of the set B is 14 times smaller than A we first enlarge the set B by a sequence of covering relations centered at x_B and $x_{B'}$, respectively – see below. Next we choose an

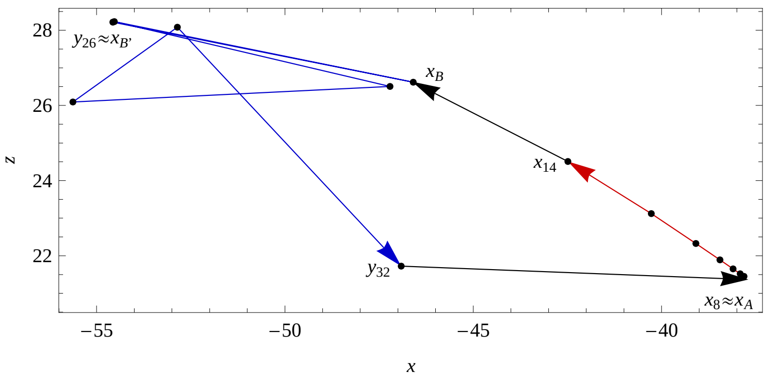

FIGURE 5. Location of the centers of h-sets on heteroclinic chains between A and B. Coordinate w is skipped since it corresponds to contracting direction.

approximate heteroclinic point inside the image of the largest (last) set in this sequence and define centers of the next sets as an approximate trajectory of this heteroclinic point. Put

$$\begin{aligned} N_i &= h(x_{B'}, L_{B'}, (1.6)^{i/2+1} r_B), \quad i = 0, 2, \ldots, 24 \\ N_{i+1} &= h(x_B, L_B, 0.95 \cdot (1.6)^{i/2+1} r_B), \quad i = 0, 2, \ldots, 24 \end{aligned}$$

Observe, that since 1.94 is an approximate eigenvalue of $DP^4(x_B)$ the factor 1.6 is chosen so we will be able to prove such expansion using rigorous numerics. Moreover, much of the expansion by the factor 1.94 is realized on the part of the periodic orbit from x_B to $x_{B'}$. Between $x_{B'}$ and x_B we observe that the weaker expansion is only slightly above one. Therefore we use the factor 0.95 to simplifying the verification of the covering relations $N_i \overset{P^2}{\Longrightarrow} N_{i+1}$ when i is an even number.

Finally, we choose an approximate heteroclinic point

$y_{26} = (-54.570284083623093352, 28.214471604985810416, 0.054984886989925152445)$

The centers of the next sets are chosen on an approximate trajectory of y_{26}, i.e.

$$y_{i+1} = P^2(y_i), \quad \text{for } i = 26, 15, \ldots, 31$$

As in the case of the heteroclinic chain from A to B the actual used coordinates of the points y_i, $i = 27, 16, \ldots, 32$ are written to the output file `centersOfSets.dat` by the program realizing the computer assisted proof. See Figure 5 where the location of these points is presented. We also choose better coordinate systems for points far from the periodic orbit B and B'. Namely,

$$\begin{aligned} L_i &= L_{B'}, \quad \text{for } i = 0, 2, \ldots, 28 \\ L_i &= L_B, \quad \text{for } i = 1, 3, \ldots, 29 \end{aligned}$$

$$L_{30} = \begin{bmatrix} 0.18506058721822621 & -0.9047729142996405 & 0.021434994608289 \\ -0.9827270967747216 & 0.42589341781181056 & 0.01066766362367 \\ 0.000179786462969468 & -0.0008776185445062 & 0.9997133298896 \end{bmatrix}$$

$$L_{31} = \begin{bmatrix} 0.11245089275337684 & -0.93803329257526924 & 0.02650407895 \\ -0.99365692790365467 & 0.34654368604561714 & 0.01317977954 \\ 0.00084044514378506187 & -0.001007810654872175 & 0.9995618176 \end{bmatrix}$$

$$L_{32} = \begin{bmatrix} -0.47800428984852406 & -0.95191504524305337 & 0.02650407895 \\ 0.8783565262524754 & 0.30635936487099347 & 0.01317977954 \\ -0.0013083103961775116 & -0.001298536005259685 & 0.9995618176 \end{bmatrix}$$

For $i = 26, \ldots, 32$ we put

$$N_i = h(y_i, L_i, r_i)$$

where

$$\begin{aligned} r_{26} = r_{27} &= 1.2 r_B, \\ r_{28} &= 2 r_B, \\ r_{29} &= 1.8 r_B, \\ r_{30} &= 3 r_B, \\ r_{31} &= \frac{5}{7} 10^{-6} (10, 100, 0.05), \\ r_{32} &= 45 \cdot 10^{-4} (4.3, 20, 1). \end{aligned}$$

Now, we can state the main lemma of this section

Lemma 14. *The following covering relations hold*

$$\begin{aligned} A \overset{P^2}{\Longrightarrow} M_0 \overset{P^2}{\Longrightarrow} M_1 \overset{P^2}{\Longrightarrow} \cdots \overset{P^2}{\Longrightarrow} M_{14} \overset{P^2}{\Longrightarrow} B, \\ B \overset{P^2}{\Longrightarrow} N_0 \overset{P^2}{\Longrightarrow} N_1 \overset{P^2}{\Longrightarrow} \cdots \overset{P^2}{\Longrightarrow} N_{32} \overset{P^2}{\Longrightarrow} A, \\ M_{14} \overset{P^2}{\Longrightarrow} N_{25} \overset{P^2}{\Longrightarrow} N_{24}, \\ N_{32} \overset{P^2}{\Longrightarrow} M_7 \overset{P^2}{\Longrightarrow} M_7. \end{aligned}$$

The existence of the covering relations asserted in Lemma 14 is proven by using a rigorous ODE solver implemented in [CAPD] and by means of the algorithms presented in [WZ2, §6]. Since the list of parameters is long we do not present it here. This list is available in the data file `gridStepOrder.dat` attached to the C++ program realizing the computer assisted proof of the above lemma.

Proof of Theorem 11: From Lemma 14 we can build the following sequences of covering relations

$$M_7 \underbrace{\overset{P^2}{\Longrightarrow} M_7 \overset{P^2}{\Longrightarrow} M_7 \overset{P^2}{\Longrightarrow} \cdots \overset{P^2}{\Longrightarrow} M_7}_{8 \text{ times}} \tag{14}$$

$$M_7 \overset{P^2}{\Longrightarrow} M_8 \overset{P^2}{\Longrightarrow} \cdots \overset{P^2}{\Longrightarrow} M_{14} \overset{P^2}{\Longrightarrow} N_{25} \tag{15}$$

$$N_{25} \overset{P^2}{\Longrightarrow} N_{26} \overset{P^2}{\Longrightarrow} \cdots \overset{P^2}{\Longrightarrow} N_{32} \overset{P^2}{\Longrightarrow} M_7 \tag{16}$$

$$N_{25} \underbrace{\overset{P^2}{\Longrightarrow} N_{24} \overset{P^2}{\Longrightarrow} N_{25} \overset{P^2}{\Longrightarrow} \cdots \overset{P^2}{\Longrightarrow} N_{24} \overset{P^2}{\Longrightarrow} N_{25}}_{4 \text{ times}} \tag{17}$$

Fix any sequence $\sigma \in \{M_7, N_{25}\}^{\mathbb{Z}}$. Let $\tau \in \{M_7, \ldots, M_{14}, N_{24}, \ldots, N_{32}\}^{\mathbb{Z}}$ be a sequence obtained form σ in the following way.

- if $(\sigma_i, \sigma_{i+1}) = (M_7, M_7)$ then $(\tau_{8i}, \ldots, \tau_{8i+7}) = \underbrace{(M_7, \ldots, M_7)}_{8 \text{ times}}$,
- $(\sigma_i, \sigma_{i+1}) = (M_7, N_{25})$ then $(\tau_{8i}, \ldots, \tau_{8i+7}) = (M_7, \ldots, M_{14})$,
- $(\sigma_i, \sigma_{i+1}) = (N_{25}, M_7)$ then $(\tau_{8i}, \ldots, \tau_{8i+7}) = (N_{25}, \ldots, N_{32})$,
- if $(\sigma_i, \sigma_{i+1}) = (N_{25}, N_{25})$ then $(\tau_{8i}, \ldots, \tau_{8i+7}) = \underbrace{(N_{25}, N_{24}, \ldots, N_{25}, N_{24})}_{4 \text{ times}}$.

From Theorem 4, (14-17) and the above we obtain that there exists a full orbit $(x_i)_{i\in\mathbb{Z}}$ with respect to P^2 and the sequence τ. Clearly, the sequence $(x_{8i})_{i\in\mathbb{Z}}$ is a full orbit with respect to P^{16} and the sequence σ. Moreover, for periodic sequences σ this full orbit may be chosen to be periodic with the same principal period. This proves that there exists an invariant set for P^{16} (consisting of these full orbits) on which P^{16} is Σ_2-chaotic.

From Lemmas 13 and 14 it follows that we can build infinitely many different sequences of covering relations satisfying the assumptions of Theorem 8 which gives infinitely many heteroclinic orbits between the unique fixed point u_A for P^2 in $|A|$ and a fixed point u_B for P^4 in $|B|$ in both directions. Moreover, we can construct infinitely many sequences of covering relations which, together with Theorem 8, prove the existence of infinitely many homoclinic orbits to u_A and to u_B.

Finally, we can easily verify, using rigorous numerics, that the trajectory of u_A intersects the Poincaré section exactly twice in disjoint sets and the trajectory of u_B intersects Poincaré section exactly four times in pairwise disjoint sets. This proves that these orbits are of period two and four, respectively. ∎

4.4. **Implementation notes.** In order to compute the Poincaré map P^2 with its partial derivatives we used the interval arithmetic [IE, Mo], the set algebra and the $\mathcal{C}^1$-Lohner algorithm [Z1] developed at the Jagiellonian University by the CAPD group [CAPD]. The C++ source files of the program with an instruction of how it should be compiled and run are available at [W].

All computations were performed on the Intel Xeon, 3GHz processor. The program has been tested under gcc-4.2.1, gcc-4.1.2 and gcc-3.4.5 both under linux and MS Windows XP Professional. The program can be compiled and run as a console application (in that case any additional libraries are not used) or as a graphical application which shows computed bounds during the computations (wxWidgets graphics library is required).

The program is also available to compile and run in a multithreaded version for multiprocessor computers. In that case computations are available only in text mode and they are as much faster as there are processors free to use for the program. We successfully compiled and ran the program on a computer with 8 Quad-Core AMD Opteron(tm) 8354, 2.2GHz processors. The programm executes almost 3 hours on this computer.

5. Final remarks.

In this paper we gave a computer assisted proof of the existence of rich dynamics, including infinitely many homo- and heteroclinic solutions for the hyperchaotic Rössler system (2). We are, however, far away from understanding the complete dynamics of the Rössler system. In fact, we did not prove either if such homoclinic and heteroclinic orbits are transversal or if there exists a hyperbolic set on which the chaotic dynamics is defined. This could be done by verifying the cone conditions for all covering relations from Lemma 14 but even if it is possible it would require a large computational time.

On the other hand, since all the methods used in the proof are robust under perturbations, the same statement as in Theorem 11 holds true for an open set of parameter values of the system (2). It could be interesting to verify if, in this range of parameter values or even for those used in this paper, the Rössler system has a stable periodic orbit (probably of a very high period).

Other interesting questions concern the existence of homoclinic tangencies or period doubling bifurcations when the parameter values vary. It seems that the method proposed in [AM] and [WZ3] could be applied to study these phenomena.

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E-mail address: `wilczak@ii.uj.edu.pl`