THE EXISTENCE OF SHILNIKOV HOMOCLINIC ORBITS IN THE MICHELSON SYSTEM: A COMPUTER ASSISTED PROOF.

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Abstract. In this paper we present a new topological tool which allows to prove the existence of Shilnikov homoclinic or heteroclinic solutions. We present an application of this method to the Michelson system $y''' + y' + 0.5y^2 = c^2$ [16]. We prove that there exists a countable set of parameter values c for which a pair of the Shilnikov homoclinic orbits to the equilibrium points $(\pm c\sqrt{2}, 0, 0)$ appear. This result was conjectured by Michelson [16]. We also show that there exists a countable set of parameter values for which there exists a heteroclinic orbit connecting the equilibrium $(-c\sqrt{2}, 0, 0)$ possessing one dimensional unstable manifold with the equilibrium $(c\sqrt{2}, 0, 0)$ possessing one dimensional stable manifold. The method used in the proof can be applied to other reversible systems.

To verify assumptions of the main topological theorem for the Michelson system we use rigorous computations based on interval arithmetic.

1. Introduction

In recent years topological tools like the Conley index [18] combined with rigorous computations based on computational homology [11] have been successfully applied to study dynamics in finite and infinite dimensions (see [19] and [8] as an example).

The aim of this paper is to present a topological tool for proving the existence of heteroclinic or homoclinic solutions for a parameterized system of finite dimensional ODEs. The method has geometrical assumptions which can be reduced to a finite number of inequalities. This enables the verification of the assumptions in computer assisted computations.

The method originates from [27, 5] where a geometric method for proving the existence of periodic orbits and chaotic dynamic for maps in $\mathbb{R}^n$ is presented. The main idea of such a method [27, 5] is to verify some kind of topological transversality called covering relations, among some subsets of the phase space which are called h-sets. The main theorem [5, Thm.4] says that if there exists a loop of covering relations $N_0 \Longrightarrow N_1 \Longrightarrow \cdots \Longrightarrow N_n = N_0$, where N_i, $i = 0, \ldots, n$ are mutually disjoint h-sets then there is a periodic point whose image belongs in turn to N_i. Moreover, such a periodic point has the same principal period as the sequence of covering relations. If there exist infinitely many different loops of covering relations then we can obtain infinitely many geometrically different associated periodic points.

Next, the ideas presented in [27, 5] have been adopted to the reversible dynamical systems [23, 24, 25, 26], where the geometric method for proving the existence of symmetric periodic, homoclinic and heteroclinic solutions is presented.

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The main topological result of this paper will be presented in Section 3 – Theorem 3.3. Let X be a topological space and let $\phi_\lambda : \mathbb{R} \times X \to X$, $\lambda \in \Lambda$ be a continuous family of local dynamical systems.

Generally speaking, Theorem 3.3 gives a geometrical tool for proving the existence of special trajectories of ϕ_λ which appear for some isolated parameters λ only, like the Shilnikov homoclinic solutions. The main idea of the method is to define covering relation from an h-set in the parameter space into the phase space of the system under consideration and next use the sequence of covering relations in the phase space. If such a sequence of covering relations is carefully chosen then we can deduce that for a certain parameter value some kind of bifurcation occurs, including codimension one or higher homoclinic or heteroclinic bifurcations.

The second part of the paper consists of an application of the method introduced in Section 3 to the Michelson system [16]

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = c^2 - y - \frac{1}{2}x^2 \end{cases} \tag{1}$$

The system (1) arises as an equation for the derivative of a traveling wave or a steady state solution of the one-dimensional Kuramoto-Sivashinsky equation

$$u_t = -u_{xxxx} - u_{xx} - uu_x$$

The system (1) possesses two equilibrium points: $x_-(c) = (-c\sqrt{2}, 0, 0)$ and $x_+(c) = (c\sqrt{2}, 0, 0)$. Moreover, it is measure preserving and $R : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$R(x, y, z, t) = (-x, y, -z, -t). \tag{2}$$

is a reversing symmetry of the system, which means that if $(x(t), y(t), z(t))$ is a solution of (1) then $(-x(-t), y(-t), -z(-t))$ is a solution too.

The dynamics of the system (1) have been studied in several papers. McCord [17] showed that for every sufficiently large c there exists a unique nonstationary bounded solution – a heteroclinic orbit connecting equilibrium points. Mrozek and Żelawski [21] proved that for $c \geq 1$ there exists a heteroclinic solution of (1) connecting the equilibrium points $x_+(c)$ and $x_-(c)$.

Lau observed [14] that when c decreases to $c_\infty \approx 1.26$ then a cascade of "cocoon" bifurcations of the two dimensional stable-unstable manifolds of the equilibria appears. At the limit point c_∞ a saddle-node periodic orbit arises. Such a cascade of heteroclinic bifurcations has been studied by Dumortier, Ibáñez and Kokubu [4] in more general context of reversible vector fields in $\mathbb{R}^3$. In that paper the authors showed that such phenomena can occur not only in the Michelson system but it is a consequence of the existence of cups-transverse heteroclinic chain connecting equilibria through the symmetric saddle-node periodic orbit.

In [23, 24] it was proved that for $c = 1$ the system is chaotic and it possesses infinitely many R-symmetric heteroclinic solutions connecting $x_+(c)$ with $x_-(c)$.

Some interesting results about the dynamics close to the Hopf-zero bifurcation in reversible vector fields in $\mathbb{R}^3$ are presented in [13]. When applied to the Michelson system they provide the existence of extremely complicated solutions including heteroclinic and homoclinic connections and chaotic dynamics for the parameter values c close to zero [13, Theorem 1.4].

In this paper we study the Michelson system (1) for the range of parameter values between $c_{\min} = 0.8285$ and $c_{\max} = 0.861$.

Theorem 1.1. *For each parameter value $c \in [c_{\min}, c_{\max}]$ the Michelson system (1) is Σ_4 chaotic, i.e. a suitable Poincaré map is semiconjugated to the full shift on four symbols.*

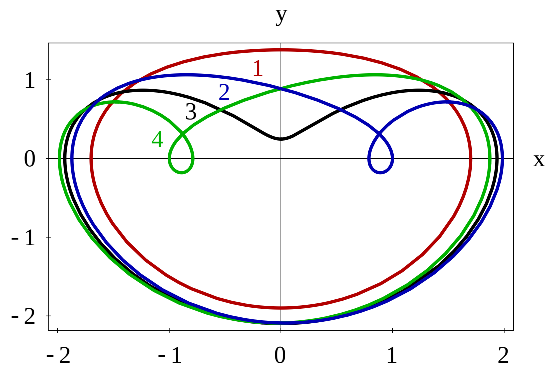

FIGURE 1. The four periodic orbits.

The symbolic dynamics are built on four periodic orbits presented in Fig. 1. These periodic orbit are "close" to each other in some neighborhood V of $(0,-2,0)$. We will show that for all parameter values $c \in [c_{\min}, c_{\max}]$ and an arbitrary sequence $(i_j)_{j\in\mathbb{Z}} \subset \{1,2,3,4\}^{\mathbb{Z}}$ there exists a solution of (1) starting from V which follows the periodic orbits in the order prescribed by the sequence $(i_j)_{j\in\mathbb{Z}}$. Moreover, we will show that if such code sequence $(i_j)_{j\in\mathbb{Z}}$ is periodic then the associated solution of (1) may be chosen to be a periodic solution.

Theorem 1.1 is not the main goal of this paper, however it helps to understand the dynamics of homoclinic and heteroclinic solutions.

Kuramoto and Tsuzuki [12] discovered an explicit formula for the heteroclinic solution connecting $x_-(c)$ with $x_+(c)$:

$$x(t) = \alpha(-9\tanh(\beta t) + 11\tanh^3(\beta t)), \tag{3}$$

where $\alpha = 15\sqrt{11/19^3}$, $\beta = \frac{1}{2}\sqrt{11/19}$ and $c = c_{KT} = \alpha\sqrt{2} \approx 0.84952$. This is a solution connecting the equilibrium $x_-(c)$ possessing one dimensional unstable manifold with the equilibrium $x_+(c)$ possessing one dimensional stable manifold. The existence of such a heteroclinic solution is in general codimension two phenomena. However, if the system is reversible and the equilibria are symmetric then the existence of one dimensional heteroclinic connection can be proven by searching for the intersection of one dimensional unstable manifold of $x_-(c)$ with the set of fixed points of the reversing symmetry – in the Michelson system it is y axis. Therefore, in that case the existence of one dimensional heteroclinic connection has codimension one.

The above heteroclinic solution (3) is the simplest of the countable heteroclinic family which exists in the range of parameter values $[c_{\min}, c_{\max}]$. We will show that for certain parameter values c one branch of the unstable manifold of the equilibrium point $x_-(c)$ can make an arbitrarily large but finite number of loops close to the periodic orbits and finally it intersects y axis. The symmetry argument implies that it must be a heteroclinic solution connecting $x_-(c)$ with $x_+(c)$. The following theorem summarizes this fact.

Theorem 1.2. *There exists a countable infinite set of parameter values $c \in [c_{\min}, c_{\max}]$ for which the system (1) possesses a heteroclinic orbit connecting $(-\sqrt{2}c, 0, 0)$ with $(\sqrt{2}c, 0, 0)$ along one dimensional unstable-stable manifolds.*

Some examples of heteroclinic orbits resulting from Theorem 1.2 are presented in Fig. 2. The above mentioned structure of heteroclinic solutions will be precisely described in Theorem 4.3.

Theorem 1.3. *For all parameter values $c \in [c_{\min}, c_{\max}]$ there exist infinitely many R-symmetric heteroclinic solutions connecting $(\sqrt{2}c, 0, 0)$ with $(-\sqrt{2}c, 0, 0)$.*

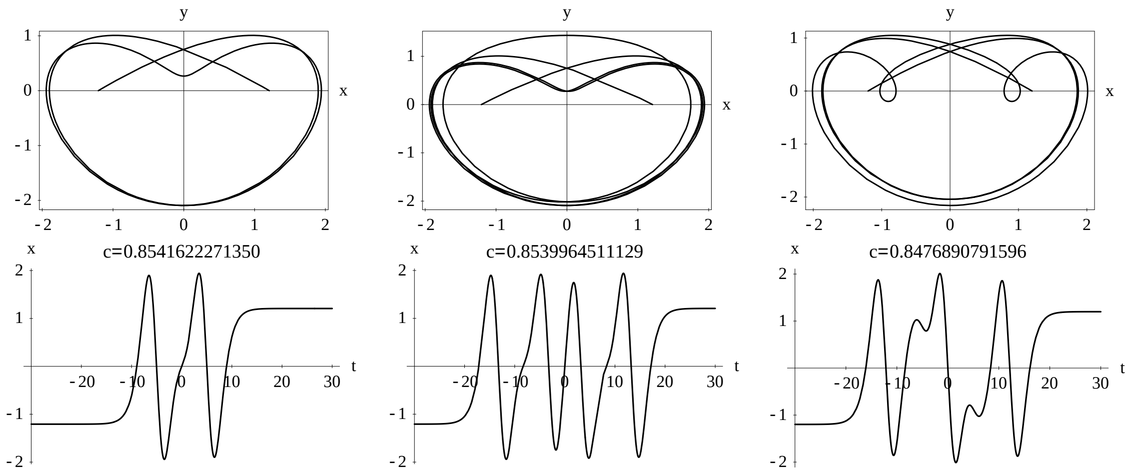

FIGURE 2. Heteroclinic orbits connecting $x_-(c)$ with $x_+(c)$ for certain parametr values.

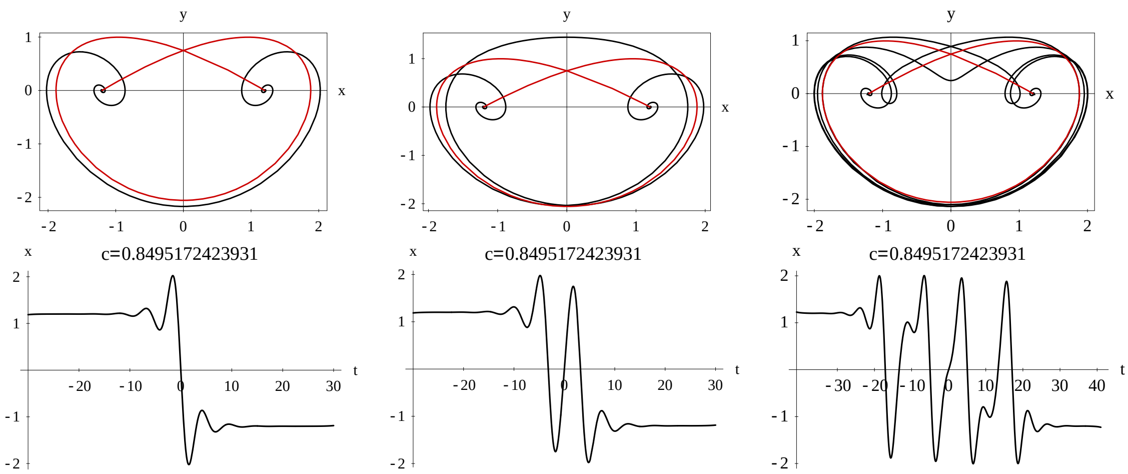

FIGURE 3. Heteroclinic loops between equilibria for the parametr value $c = c_{KT}$. The solution (3) found by Kuramoto and Tsuzuki [12] is in red color.

Some examples of heteroclinic loops for the parameter value $c = c_{KT}$ are presented in Fig. 3.

Remark 1.4. *It should be noted that the proofs of Theorems 1.2 and 1.3 use the fact that the Michelson system is reversible. Since the method used in the proof of Theorem 1.3 has the assumptions which are open (i.e. they can be reduced to finite number of strong inequalities) the existence of symmetric heteroclinic solutions connecting* $x_+(c)$ *with* $x_-(c)$ *persists under small reversible perturbations. Nevertheless, the method presented in this paper does not give any information about the transversality of such heteroclinic orbits.*

The proof of Theorem 1.3 is almost the same as of the similar result for parameter $c = 1$ [24, Thm.3.1]. The main reason for which we present Theorem 1.3 is that it shows together with Theorem 1.2 that for a countable set of parameter values there exist infinitely many heteroclinic connections between equilibrium points in both directions called Bykov cycles. This provides a Shilnikov like structure [9, 10].

Remark 1.5. *The existence of reversible and transverse heteroclinic cycle for the parameter value $c = c_{KT}$ has been proven in* [7]. *The existence of a countable family of Bykov cycles for the Michelson system follows from the existence of one Bykov cycle for the parameter value $c = c_{KT}$ - see* [3]. *In this paper we present an alternative proof for such result based on a geometric approach.*

The existence of a reversible Bykov cycle has important dynamical consequences. Dumortier, Ibáñez and Kokubu [4] have shown that if a Bykov cycle appears for some parameter value λ_∞ then there are infinite sequences $\lambda_n^- < \lambda_\infty < \lambda_n^+$ converging to λ_∞ as $n \to \infty$ such that for each $\lambda_n^\pm$ a periodic saddle-node bifurcation occurs.

Lamb, Teixeira and Webster [13] proved that if such a heteroclinic loop appears in a reversible system in $\mathbb{R}^3$ then, generically, a countable infinite set of symmetric and asymmetric hetereroclinic orbits accumulating to the heteroclinic cycle exists. Also, there exists a countable infinite set of periodic orbits converging to the heteroclinic cycle as their period goes to infinity. Note that these periodic solutions are different from those referred to in Theorem 1.1 – in fact they are close to the heteroclinic cycle while the periodic orbits resulting from Theorem 1.1 are separated from the equilibrium points.

Theorem 1.6. *There exists a countable infinite set of parameter values $C_h \subset [c_{\min}, c_{\max}]$ for which the Michelson system (1) possesses a pair of homoclinic orbits to the equilibrium points.*

Some examples of homoclinic orbits resulting from Theorem 1.6 are presented in Fig. 4 (see also [16, Figs. 4a, 4b, 4c]).

Remark 1.7. *In* [7, 3] *it has been shown that the existence of a Bykov cycle in the Michelson system for the parameter value $c = c_{KT}$ implies the existence of a Shilnikov homoclinic solution in each neighborhood of the parameter value $c = c_{KT}$. The main difference in the approach presented in this paper is that we do not require reversibility of the system as well as the existence of a reversible Bykov cycle. Hence, the method can be applied to the other systems which are not reversible.*

We will show that for certain parameter values c one branch of the unstable manifold of the equilibrium point $x_-(c)$ leaves the point, stays close to the four periodic orbits for an arbitrarily large but finite time, and then tends back to the equilibrium point. This gives a Shilnikov homoclinic solution. The existence of homoclinic solutions to $x_+(c)$ follows from the symmetry property of (1). The structure of homoclinic orbits will be precisely described in Theorem 4.4. The set C_h has a very complicated structure. One can prove that the set of accumulation points of C_h forms a Cantor set and for these parameter values the unstable manifold of $x_-(c)$ stays all the time close to the periodic orbits.

The above applications of the method introduced in Section 3 to the Michelson system have been proved with a computer assistance. In Section 5 we present the algorithms and we explain the main ideas of the computer assisted part. In Section 4 we present a more precise statement of Theorems 1.6, 1.2 and 1.3 in the terms of admissible sequences for a suitable Poincaré map. There we also present the proofs of these results. In Section 2 we recall main definitions and properties of covering relations.

2. Topological tools: h-sets and covering relations

In this section we present main topological tools used in this paper. The crucial notion is that of *covering relation* [5].

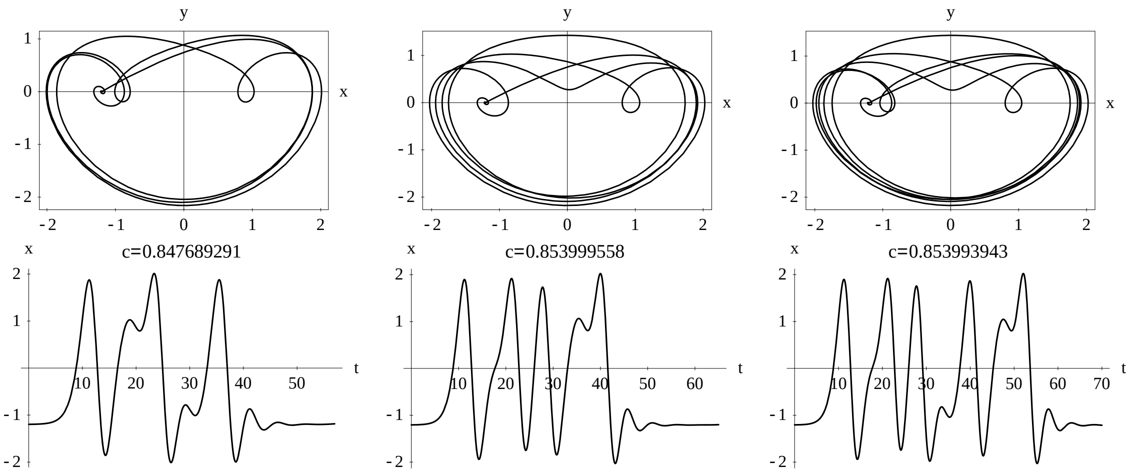

FIGURE 4. The Shilnikov homoclinic orbits to $x_-(c)$ for certain parameter values.

2.1. **h-sets. Notation:** For a given norm in $\mathbb{R}^n$ by $B_n(c,r)$ we denote an open ball of radius r centered at $c \in \mathbb{R}^n$. When the dimension n is obvious from the context we will drop the subscript n. By B_n we will denote the unit ball $B_n(0,1)$. We set $\mathbb{R}^0 := \{0\}$, $B_0(0,r) := \{0\}$, $\partial B_0(0,r) := \emptyset$.

For a given set Z, by int Z, $\overline{Z}$, ∂Z we denote the interior, the closure and the boundary of Z, respectively. For a map $h : [0,1] \times Z \to \mathbb{R}^n$ we set $h_t := h(t,\cdot)$. By Id we denote the identity map. For a map f, by dom(f) we denote the domain of f. For an open subset $N \subset \mathbb{R}^n$ and $c \in \mathbb{R}^n$ by $\deg(f,N,c)$ we denote the local Brouwer degree [2] (see also Appendix in [5]).

Definition 2.1. [5, Definition 1] *An h-set, N, is a quadruple $(|N|, u(N), s(N), c_N)$ such that*

- $|N|$ *is a compact subset of* $\mathbb{R}^n$
- $u(N), s(N) \in \{0,1,2,\dots\}$ *are such that* $u(N) + s(N) = n$
- $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ *is a homeomorphism such that*

$$c_N(|N|) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

We set

$$\begin{aligned}
\dim(N) &:= n,\\
N_c &:= \overline{B_{u(N)}} \times \overline{B_{s(N)}},\\
N_c^- &:= \partial B_{u(N)} \times \overline{B_{s(N)}},\\
N_c^+ &:= \overline{B_{u(N)}} \times \partial B_{s(N)},\\
N^- := c_N^{-1}(N_c^-), &\quad N^+ = c_N^{-1}(N_c^+).
\end{aligned}$$

Hence an h-set, N, is a product of two closed balls in some coordinate system. The numbers $u(N)$ and $s(N)$ are called the nominally unstable and nominally stable dimensions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Observe that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$.

2.2. **Covering relations.** Here we present a modification of [5, Definition 6]. The main difference is that we do not require for the 'stable' dimensions of h-sets N, M to be equal.

Definition 2.2. *Assume that N, M are h-sets, such that $u(N) = u(M) = u$ and let $f : N \to \mathbb{R}^{\dim(M)}$ be a continuous. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$. We say that N f-covers M, denoted by*

$$N \overset{f}{\Longrightarrow} M$$

if the following two conditions are satisfied

1.: *there exists a homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$ such that*

$$h_0 = f_c, \tag{4}$$
$$h([0,1], N_c^-) \cap M_c = \emptyset, \tag{5}$$
$$h([0,1], N_c) \cap M_c^+ = \emptyset. \tag{6}$$

2.: *There exists a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$ such that*

$$h_1(p, q) = (A(p), 0), \text{ for } p \in \overline{B_u} \text{ and } q \in \overline{B_{s(N)}}, \tag{7}$$
$$A(\partial B_u) \subset \mathbb{R}^u \setminus \overline{B_u}. \tag{8}$$

Intuitively, $N \overset{f}{\Longrightarrow} M$ if f stretches N in the nominally unstable direction, so that its projection onto the nominally unstable direction in M covers in topologically nontrivial manner the projection of M. In the nominally stable direction N is contracted by f. As a result N is mapped across M in the unstable direction, without touching M^+. The geometry of this concept is presented in Fig. 5.

The covering relation between the sets of unequal dimensions plays an important role when we consider a dynamical system depending on some parameter value. The main idea of the method presented in this paper is to introduce the notion of covering relations from the parameter space into the phase space of the system under consideration. Clearly, the dimension of the parameter space may be different than the dimension of the phase space. It should be observed that the topological information resulting from the existence of covering relation $N \overset{f}{\Longrightarrow} M$ is included in condition (8) which implies that an associated linear map A is an isomorphism and therefore the Brouwer degree $\deg(A, B_u, 0) \neq 0$.

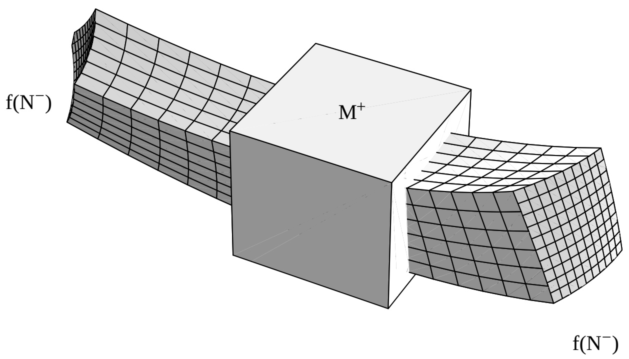

FIGURE 5. An example of covering relation. In this case $u(N) = u(M) = 1$ and $s(N) = s(M) = 2$.

Definition 2.3. *Let $\{M_1, \ldots, M_n\}$ be a collection of h-sets in $\mathbb{R}^D$ and let $f : \bigcup_{i=1}^n |M_i| \longrightarrow \mathbb{R}^D$ be continuous. We say that the sequence $(i_0, i_1, \ldots, i_k)$ is admissible with respect to f if*

$$M_{i_0} \overset{f}{\Longrightarrow} M_{i_1} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} M_{i_k}.$$

Definition 2.4. *Let $\{M_1, \ldots, M_n\}$ be a collection of h-sets in $\mathbb{R}^D$ and let $f : \bigcup_{i=1}^n |M_i| \longrightarrow \mathbb{R}^D$ be continuous. We say that the sequence $(i_j)_{j\in\mathbb{Z}} \in \{1, \ldots, n\}^{\mathbb{Z}}$ is admissible with respect to f if*

$$M_{i_j} \overset{f}{\Longrightarrow} M_{i_{j+1}}, \quad \text{for } j \in \mathbb{Z}.$$

The following theorem is one of the basic results in the covering relations method.

Theorem 2.5 ([5, Theorem 9, Corollary 12])**.** *Let $\{M_1, \ldots, M_n\}$, $n > 0$ be a collection of h-sets in $\mathbb{R}^D$ and let $f : \bigcup_{i=1}^n |M_i| \longrightarrow \mathbb{R}^D$ be continuous. If $(i_j)_{j\in\mathbb{Z}}$ is an admissible sequence with respect to f then there exists a sequence $(x_j)_{j\in\mathbb{Z}}$, such that*

$$\begin{aligned} x_j &\in \operatorname{int}|M_{i_j}|, \quad \text{for } j \in \mathbb{Z}, \\ f(x_j) &= x_{j+1}, \quad \text{for } j \in \mathbb{Z}. \end{aligned}$$

Moreover, if the sequence $(i_j)_{j\in\mathbb{Z}}$ is periodic with the principal period p and the sets M_i are mutually disjoint then the sequence $(x_j)_{j\in\mathbb{Z}}$ may be chosen to be a periodic orbit of f with the principal period p.

Observe that in the case when f is injective, the backward trajectory of x_0 is uniquely defined and $f^j(x_0) \in \operatorname{int}|M_{i_j}|$ for $j \in \mathbb{Z}$.

3. The topological theorem

In this section we present the main topological tool used in the proof of Theorems 1.6, 1.2 and 1.3. We first introduce the following definitions.

Definition 3.1. *Let N be an h-set. Let $b : \overline{B_{u(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is a horizontal disk in N if there exists a homotopy $h : [0,1] \times \overline{B_{u(N)}} \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c && (9) \\ h_1(x) &= (x, 0), \qquad \text{for all } x \in \overline{B_{u(N)}} && (10) \\ h(t,x) &\in N_c^-, \qquad \text{for all } t \in [0,1] \text{ and } x \in \partial B_{u(N)} && (11) \end{aligned}$$

Definition 3.2. *Let N be an h-set. Let $b : \overline{B_{s(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is a vertical disk in N if there exists a homotopy $h : [0,1] \times \overline{B_{s(N)}} \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c \\ h_1(x) &= (0, x), \qquad \text{for all } x \in \overline{B_{s(N)}} \\ h(t,x) &\in N_c^+, \qquad \text{for all } t \in [0,1] \text{ and } x \in \partial B_{s(N)}. && (12) \end{aligned}$$

The geometry of these definitions is presented in Fig. 6. In this case the vertical disc is a curve ($s(N) = 1$) which can be deformed into a line connecting center points of the top and bottom walls. Moreover, the end points of this curve belong to N^+ throughout this deformation. Similarly, the horizontal disc is a surface ($u(N) = 2$) whose boundary is contained in N^- and which can be deformed into the horizontal plane. Moreover, the boundary of this surface cannot leave N^- throughout this deformation.

Notice, that if $u(N) = s(N)$ then we can find a disc which is both horizontal and vertical in N – see Fig. 7. Obviously, the required homotopies in the Definitions 3.1 and 3.2 are different.

The following theorem is the main topological result of this paper.

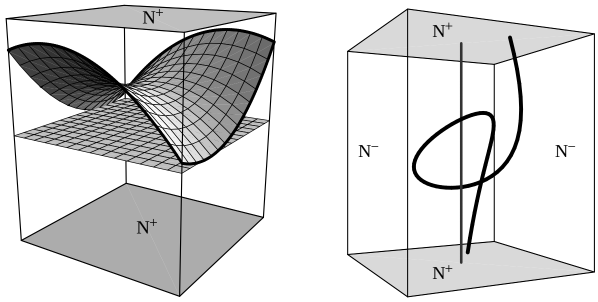

FIGURE 6. A horizontal disc in an h-set N with $u(N) = 2$ and $s(N) = 1$ (left). A vertical disc in an h-set N with $u(N) = 2$ and $s(N) = 1$ (right).

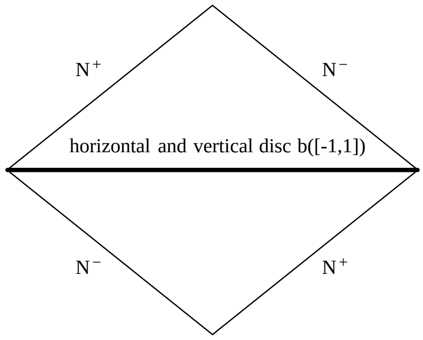

FIGURE 7. A curve b forms both horizontal and vertical discs in N.

Theorem 3.3. *Let Z, $N_0, N_1, \dots, N_k$, for $k > 0$ be h-sets such that $\dim(Z) = u(Z) = u(N_0) = \dots = u(N_k)$. Assume $W : |Z| \to \mathbb{R}^{\dim(N_0)}$, $f_i : |Z| \times |N_i| \to \mathbb{R}^{\dim(N_{i+1})}$, $i = 0, \dots, k-1$ are continuous. Let $v : \overline{B_{s(N_k)}} \to |N_k|$ be a vertical disc in N_k.*

If $Z \overset{W}{\Longrightarrow} N_0$ and for every $z \in |Z|$

$$N_0 \overset{f_0(z,\cdot)}{\Longrightarrow} N_1 \overset{f_1(z,\cdot)}{\Longrightarrow} \dots \overset{f_{k-1}(z,\cdot)}{\Longrightarrow} N_k$$

then there exists $z_0 \in |Z|$ such that

$$\begin{aligned} W(z_0) &\in |N_0|, \\ (f_i(z_0,\cdot) \circ \dots \circ f_0(z_0,\cdot))(W(z_0)) &\in |N_{i+1}|, \quad \text{for } i = 0, \dots, k-2, \\ (f_{k-1}(z_0,\cdot) \circ \dots \circ f_0(z_0,\cdot))(W(z_0)) &\in v(B_{s(N_k)}). \end{aligned} \tag{13}$$

Before we present the proof of the Theorem 3.3 let us prove the following corollary.

Corollary 3.4. *Let Z, N_i, $i \in \mathbb{N}$ be h-sets such that $\dim(Z) = u(Z) = u$ and $u(N_i) = u$ for $i \in \mathbb{N}$. Let $W : |Z| \to \mathbb{R}^{\dim(N_0)}$, $f_i : |Z| \times |N_i| \to \mathbb{R}^{\dim(N_{i+1})}$ for $i \in \mathbb{N}$.*

If $Z \overset{W}{\Longrightarrow} N_0$ and for every $z \in |Z|$ and $i \in \mathbb{N}$

$$N_i \overset{f_i(z,\cdot)}{\Longrightarrow} N_{i+1}$$

then there exists $\bar{z} \in |Z|$ such that

$$\begin{aligned} W(\bar{z}) &\in |N_0|, \\ (f_i(\bar{z},\cdot) \circ \dots \circ f_0(\bar{z},\cdot))(W(\bar{z})) &\in |N_{i+1}|, \quad \text{for } i \in \mathbb{N}. \end{aligned} \tag{14}$$

Proof: Let us fix $k \in \mathbb{N}$. Let

$$v_k : \overline{B_{s(N_k)}} \ni x \to c_{N_k}^{-1}(0, x) \in |N_k|$$

be a vertical disc in N_{k+1} (notice that it is a well defined vertical disc for $s(N_k) = 0$ too). Now Theorem 3.3 implies that for every $k \in \mathbb{N}$ there exists a point $z_k \in |Z|$ such that $W(z_k) \in |N_0|$ and

$$(f_i(z_k, \cdot) \circ \cdots \circ f_0(z_k, \cdot))(W(z_k)) \in |N_{i+1}|, \quad \text{for } i = 0, 1, \ldots, k-1.$$

Since $|Z|$ is a compact set we can find an accumulation point $\bar{z}$ of the sequence $\{z_k\}_{k\in\mathbb{Z}} \subset |Z|$. Clearly $\bar{z}$ satisfies (14). □

Let us explain the meaning of the objects which appear in Theorem 3.3. Assume we have a dynamical system depending on some parameter z. The set $|Z|$ is some subset of the parameter space of the system. Assume that we can control some property of this system with respect to the parameter z by a map W (for example the intersection of the one dimensional unstable manifold of the equilibrium with the Poincaré section). Assume $f_i(z, \cdot)$ are Poincaré maps or time flow translations. In this situation Theorem 3.3 or Corollary 3.4 say that for a sequence of covering relations of an arbitrary length we can find a parameter $z_0 \in |Z|$ such that the trajectory of $W(z_0)$ intersects in turn the sets $|N_i|$. Moreover, z_0 can be chosen in such a way that the trajectory of $W(z_0)$ (for example the unstable manifold in the system with the parameter z_0), reaches the vertical disc v in some N_k. This property will be used in the proof of the existence of symmetric heteroclinic orbits in Theorem 1.2.

Proof of Theorem 3.3: Let us recall that N_c denotes the set N in the coordinate system c_N, i.e. $N_c = \overline{B_{u(N)}} \times \overline{B_{s(N)}}$ and if $N \overset{f}{\Longrightarrow} M$ then $f_c : N_c \to \mathbb{R}^{\dim(M)}$ is defined as $f_c = c_M \circ f \circ c_N^{-1}$. To simplify the notation put $d_i := \dim(N_i)$. Put

$$\Pi := Z_c \times N_{0,c} \times N_{1,c} \times \cdots \times N_{k-1,c} \times \overline{B_{s(N_k)}} \subset \mathbb{R}^{u+d_0+\cdots+d_{k-1}+s(N_k)}.$$

We define the map $F : \Pi \to \mathbb{R}^D$, where $D = u + d_0 + \cdots + d_{k-1} + s(N_k) = d_0 + \ldots + d_k$ by

$$F\begin{bmatrix} z \\ x_0 \\ x_1 \\ \vdots \\ x_{k-1} \\ \tau \end{bmatrix} = \begin{bmatrix} x_0 - W_c(z) \\ x_1 - (f_0(c_Z^{-1}(z), \cdot))_c(x_0) \\ x_2 - (f_1(c_Z^{-1}(z), \cdot))_c(x_1) \\ \vdots \\ x_{k-1} - (f_{k-2}(c_Z^{-1}(z), \cdot))_c(x_{k-2}) \\ v(\tau) - (f_{k-1}(c_Z^{-1}(z), \cdot))_c(x_{k-1}) \end{bmatrix} \in \mathbb{R}^{d_0+d_1+\cdots+d_k} \tag{15}$$

Observe that if the homeomorphisms in the definitions of h-sets are identities, i.e. $N_{i,c} = |N_i|$, $Z_c = |Z|$ then (15) becomes simply

$$F\begin{bmatrix} z \\ x_0 \\ x_1 \\ \vdots \\ x_{k-1} \\ \tau \end{bmatrix} = \begin{bmatrix} x_0 - W(z) \\ x_1 - f_0(z, x_0) \\ x_2 - f_1(z, x_1) \\ \vdots \\ x_{k-1} - f_{k-2}(z, x_{k-2}) \\ v(\tau) - f_{k-1}(z, x_{k-1}) \end{bmatrix} \in \mathbb{R}^{d_0+d_1+\cdots+d_k}$$

It is obvious that solving (13) is equivalent to finding zeros of F in Π. We will show that the Brouwer degree of F on Π is well defined and nonzero. For this end we will construct a homotopy connecting F with some map which has nonzero degree.

Lemma 3.5. *Let N, M be h-sets such that $u = u(N) = u(M)$. Let Z be an h-set,* $\dim(Z) = k \geq 0$ *and let $f : |Z| \times |N| \to \mathbb{R}^{\dim(M)}$ be continuous. If $N \overset{f(z,\cdot)}{\Longrightarrow} M$ for $z \in |Z|$ then there exist a homotopy $h : [0,1] \times \overline{B_k} \times N_c \to \mathbb{R}^{\dim(M)}$ and a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$h(0,z,x) = \left(f(c_Z^{-1}(z),\cdot)\right)_c(x), \text{ for } z \in \overline{B_k},\ x \in N_c, \tag{16}$$

$$h([0,1],\overline{B_k},N_c^-) \cap M_c = \emptyset, \tag{17}$$

$$h([0,1],\overline{B_k},N_c) \cap M_c^+ = \emptyset, \tag{18}$$

and

$$\begin{aligned} h(1,z,(p,q)) = (A(p),0),\ \text{for } p \in \overline{B_u},\ q \in \overline{B_{s(N)}},\ z \in \overline{B_k} \\ A(\partial B_u) \subset \mathbb{R}^u \setminus \overline{B_u}. \end{aligned} \tag{19}$$

Proof: We will define the homotopy h as a homotopy superposition of two independent homotopies. First we deform the parameters $\overline{B_k}$ to zero, then we use the homotopy from the definition of covering relations for $z = 0$. More precisely, let $h_1 : [0,1] \times \overline{B_k} \times N_c \to \mathbb{R}^{\dim(M)}$ be defined by

$$h_1(t,z,x) = \left(f\left(c_Z^{-1}((1-t)z),\cdot\right)\right)_c(x). \tag{20}$$

Since $N \overset{f(z,\cdot)}{\Longrightarrow} M$ for $z \in |Z|$ and $c_Z^{-1}((1-t)z) \in |Z|$ for $t \in [0,1]$, h_1 satisfies (17–18) by (4–6).

Let $h_2 : [0,1] \times N_c \to \mathbb{R}^{s(M)}$ and $A : \mathbb{R}^u \to \mathbb{R}^u$ be the homotopy and the linear map from the definition of the covering relation for $N \overset{f(c_Z^{-1}(0),\cdot)}{\Longrightarrow} M$. Put

$$h(t,z,x) = \begin{cases} h_1(2t,z,x), & t \in \left[0,\frac{1}{2}\right] \\ h_2(2t-1,x), & t \in \left(\frac{1}{2},1\right] \end{cases}$$

The homotopy h is continuous because $h_1(\frac{1}{2},z,x) = h_2(\frac{1}{2},x) = \left(f(c_Z^{-1}(0),\cdot)\right)_c(x)$. From (20)

$$h(0,z,x) = h_1(0,z,x) = (f(c_Z^{-1}(z),\cdot))_c(x), \quad \text{for} \quad z \in B_k, x \in N_c,$$

hence (16) holds. The homotopy h satisfies (17–18) because h_1 and h_2 do. Finally

$$h(1,z,(p,q)) = h_2(1,(p,q)) = (A(p),0) \quad \text{for} \quad p \in \overline{B_u}, q \in \overline{B_{s(N)}}, z \in \overline{B_k}$$

hence (19) holds. □

Now we continue the proof of Theorem 3.3. Let

$$h_i : [0,1] \times \overline{B_u} \times N_{i,c} \to \mathbb{R}^{d_{i+1}}, \quad i = 0,1,\ldots,k-1$$

be homotopies resulting from Lemma 3.5 for covering relations $N_i \overset{f_i(z,\cdot)}{\Longrightarrow} N_{i+1}$, respectively. Let

$$h_p : [0,1] \times Z_c \to \mathbb{R}^{d_0}$$

be a homotopy from the definition of the covering relation $Z \overset{W}{\Longrightarrow} N_0$ and let

$$h_v : [0,1] \times \overline{B_{s(N_k)}} \to N_{i_k,c}$$

be a homotopy from the definition of vertical disc v in N_k. Let us define $H : [0,1] \times \Pi \to \mathbb{R}^D$ by

$$H(t,z,x_0,x_1,\ldots,x_{k-1},\tau) = \begin{bmatrix} x_0 - h_p(t,z) \\ x_1 - h_0(t,z,x_0) \\ x_2 - h_1(t,z,x_1) \\ \vdots \\ x_{k-1} - h_{k-2}(t,z,x_{k-2}) \\ h_v(t,\tau) - h_{k-1}(t,z,x_{k-1}) \end{bmatrix}$$

Notice, that $H(0,\cdot) = F$. The assertion of the theorem is the consequence of the following lemmas.

Lemma 3.6. *The Brouwer degree* $\deg(H(t,\cdot), \mathrm{int}\,\Pi, 0)$ *is well defined for* $t \in [0,1]$ *and does not depend on* t. *In other words, for all* $t \in [0,1]$ *we have*

$$\deg(H(t,\cdot), \mathrm{int}\,\Pi, 0) = \deg(H(1,\cdot), \mathrm{int}\,\Pi, 0)$$

Lemma 3.7.

$$|\deg(H(1,\cdot), \mathrm{int}\,\Pi, 0)| = 1$$

We continue the proof of Theorem 3.3. From Lemma 3.6 and Lemma 3.7 we get

$$|\deg(F, \mathrm{int}\,\Pi, 0)| = |\deg(H(0,\cdot), \mathrm{int}\,\Pi, 0)| = |\deg(H(1,\cdot), \mathrm{int}\,\Pi, 0)| = 1,$$

hence there exists $x = (z, x_0, x_1, \ldots, x_{k-1}, \tau) \in \mathrm{int}\,\Pi$ such that $F(x) = 0$. Obviously $z_0 = (c_Z)^{-1}(z) \in |Z|$ satisfies assertion of the theorem. □

Proof of Lemma 3.6: By the properties of the Brouwer degree it is sufficient to show that

$$H(t,x) \neq 0, \quad \text{for all } t \in [0,1], \text{ and } x \in \partial\Pi.$$

Take

$$x = (z, x_0, x_1, \ldots, x_{k-1}, \tau) \in \partial\Pi = \partial\left(Z_c \times N_{0,c} \times N_{1,c} \times \cdots \times N_{k-1,c} \times \overline{B_{s(N_k)}}\right)$$

Recall that $Z_c = \overline{B_u}$ and $N_{i,c} = \overline{B_u} \times \overline{B_{s(N_i)}}$ for $i = 0, 1, \ldots, k-1$. Then x satisfies at least one of the following conditions

$$z \in \partial B_u = \partial Z_c, \tag{21}$$

$$x_i \in \partial B_u \times \overline{B_{s(N_i)}} = N^-_{i,c}, \quad \text{for some } i = 0, 1, \ldots, k-1, \tag{22}$$

$$x_i \in \overline{B_u} \times \partial B_{s(N_i)} = N^+_{i,c}, \quad \text{for some } i = 0, 1, \ldots, k-1, \tag{23}$$

$$\tau \in \partial B_{s(N_k)}. \tag{24}$$

Consider case (21). Since $z \in \partial B_u$ and $Z \overset{W}{\Longrightarrow} N_0$ then from (5) it follows that $h_p(t,z) \notin N_{0,c}$ for $t \in [0,1]$. Hence $H(t,x)$ cannot be zero on the first coordinate.

Consider case (22). Since $x_i \in N^-_{i,c}$ for some $i = 0, \ldots, k-1$ and $N_i \overset{f_i(c,\cdot)}{\Longrightarrow} N_{i+1}$, from (17) we get that $h_i(t,z,x_i) \notin N_{i+1,c}$ for $t \in [0,1]$. This proves that if $i \in \{0, 1, \ldots, k-2\}$ then $h_i(t,z,x_i) - x_{i+1} \neq 0$ for $t \in [0,1]$ and if $i = k-1$ then $h_{k-1}(t,z,x_{k-1}) - h_v(t,\tau) \neq 0$ for $t \in [0,1]$.

Consider case (23). Then $x \in N^+_{i,c}$ for some $i = 0, \ldots, k-1$. If $i = 0$ then condition (6) for $Z \overset{W}{\Longrightarrow} N_0$ implies that $h_p(t,z) \notin N^+_{0,c}$, for $z \in Z_c$ and $t \in [0,1]$. This proves that $h_p(t,z) \neq x_0$ for $z \in Z_c$ and $t \in [0,1]$ and consequently $H(t,x) \neq 0$.

If $i = 1, \ldots, k-1$ then from (18) we get $h_{i-1}(t,z,x_{i-1}) \notin N^+_{i,c}$ for all $x_{i-1} \in N_{i-1,c}$, $z \in Z_c$ and $t \in [0,1]$, hence $h_i(t,z,x_{i-1}) \neq x_i$ and consequently $H(t,x) \neq 0$.

Consider case (24). From (18) it follows that $h_{k-1}(t,z,x_{k-1}) \notin N^+_{k,c}$. Since v is a vertical disc in N_k from (12) we get $h_v(t,\tau) \in N^+_{k,c}$ for all $t \in [0,1]$. Hence $H(t,x)$ cannot be zero on the last coordinate.

We have shown that $0 \notin H([0,1], \partial\Pi)$. The assertion of the lemma is the consequence of the homotopy property of the Brouwer degree. □

Proof of Lemma 3.7: Let us represent $x \in \Pi$ as

$$x = (z, p_0, q_0, \ldots, p_{k-1}, q_{k-1}, \tau) \in \mathbb{R}^u \times \mathbb{R}^{u(N_0)} \times \mathbb{R}^{s(N_0)} \times \cdots \times \mathbb{R}^{u(N_{k-1})} \times \mathbb{R}^{s(N_{k-1})} \times \mathbb{R}^{s(N_k)}.$$

Let A be a linear map from the definition of the covering relation $Z \overset{W}{\Longrightarrow} N_0$ and let A_i, $i = 0, 1, \ldots, k-1$ be linear maps resulting from Lemma 3.5 for covering relations $N_i \overset{f(z,\cdot)}{\Longrightarrow} N_{i+1}$, respectively. Consider the homotopy $C : [0,1] \times \Pi \to \mathbb{R}^D$ given by

$$C(t,x) = C\begin{bmatrix} t \\ z \\ (p_0, q_0) \\ \vdots \\ (p_{k-1}, q_{k-1}) \\ \tau \end{bmatrix} = \begin{bmatrix} ((1-t)p_0 - A(z), q_0) \\ ((1-t)p_1 - A_0(p_0), q_1) \\ ((1-t)p_2 - A_1(p_1), q_2) \\ \vdots \\ ((1-t)p_{k-1} - A_{k-2}(p_{k-2}), q_{k-1}) \\ (-A_{k-1}(p_{k-1}), \tau) \end{bmatrix}$$

Observe that $C(0,\cdot) = H(1,\cdot)$.

Lemma 3.8. *The Brouwer degree* $\deg(C(t,\cdot), \mathrm{int}\,\Pi, 0)$ *is well defined for all* $t \in [0,1]$ *and does not depend on* t. *In particular*

$$\deg(C(0,\cdot), \mathrm{int}\,\Pi, 0) = \deg(C(1,\cdot), \mathrm{int}\,\Pi, 0).$$

Proof: As in the proof of Lemma 3.6 it is sufficient to show that $0 \notin C([0,1], \partial\Pi)$. Let us fix $x = (z, p_0, q_0, \ldots, p_{k-1}, q_{k-1}, \tau) \in \partial\Pi$. Then x satisfies at least one of the following conditions

$$z \in \partial B_u = \partial Z_c, \tag{25}$$

$$p_i \in \partial B_u, \quad \text{for some } i = 0, 1, \ldots, k-1, \tag{26}$$

$$q_i \in \partial B_{s(N_i)}, \quad \text{for some } i = 0, 1, \ldots, k-1, \tag{27}$$

$$\tau \in \partial B_{s(N_k)}. \tag{28}$$

Consider case (25). If $z \in \partial B_u$ then from (8) we know that $||A(z)|| > 1$. Therefore

$$||A(z)|| > 1 \geq ||(1-t)p_0||.$$

This shows that $(1-t)p_0 - A(z) \neq 0$ and consequently $C(t,x) \neq 0$.

Consider case (26). If $p_i \in \partial B_u$ for some $i = 0, 1, \ldots, k-1$ then from (19) we get that $||A_i(p_i)|| > 1$. Therefore

$$||A_i(p_i)|| > 1 \geq ||(1-t)p_{i+1}||.$$

This shows that $(1-t)p_{i+1} - A_i(p_i) \neq 0$ and consequently $C(t,x) \neq 0$.

Consider case (27). If $q_i \in \partial B_{s(N_i)}$ for some $i = 0, 1, \ldots, k-1$ then $C(t,x)$ is nonzero because $C(t,\cdot)$ does not modify the q_i coordinate for all $t \in [0,1]$.

Finally consider case (28). Then $C(t,x)$ is nonzero, because $C(t,\cdot)$ is the identity on the τ coordinate.

Thus $0 \notin C([0,1], \partial\Pi)$ and the assertion is proved. □

Now we continue the proof of Lemma 3.7. Since $C(0,\cdot) = H(1,\cdot)$, Lemma 3.8 implies that

$$\deg(H(1,\cdot), \mathrm{int}\,\Pi, 0) = \deg(C(0,\cdot), \mathrm{int}\,\Pi, 0) = \deg(C(1,\cdot), \mathrm{int}\,\Pi, 0).$$

Let us compute the degree of $C(1,\cdot)$. From the definition of C we have

$$C(1,x) = C\begin{bmatrix} 1 \\ z \\ (p_0,q_0) \\ \vdots \\ (p_{k-1},q_{k-1}) \\ \tau \end{bmatrix} = \begin{bmatrix} (-A(z),q_0) \\ (-A_0(p_0),q_1) \\ (-A_1(p_1),q_2) \\ \vdots \\ (-A_{k-2}(p_{k-2}),q_{k-1}) \\ (-A_{k-1}(p_{k-1}),\tau) \end{bmatrix}$$

By (8) and (19) A and A_i are isomorphisms. This implies that $C(1,\cdot)$ is an isomorphism too. Recall that for a linear map L and an open set containing zero $\deg(L,D,0) = \mathrm{sgn}(\det L)$ (see Appendix in [5]). This shows that

$$|\deg(H(1,\cdot),\mathrm{int}\,\Pi,0)| = |\deg(C(1,\cdot),\mathrm{int}\,\Pi,0)| = 1.$$

The proof of Lemma 3.7 and Theorem 3.3 is complete. □

The next theorem is an extension of [25, Thm.3] to the case of unequal dimensions of h-sets which appear in the sequence of covering relations.

Theorem 3.9. *Let N_i, $i = 0,1,\dots,k$ be h-sets such that $u(N_i) = u$ for $i = 0,1,\dots,k$ and let $f_i : |N_i| \to \mathbb{R}^{\dim(N_{i+1})}$, $i = 0,1,\dots,k-1$ be continuous. Let $b : \overline{B_u} \to |N_0|$ be a horizontal disc in N_0 and let $v : \overline{B_{s(N_k)}} \to |N_k|$ be a vertical disc in N_k. If $N_i \overset{f_i}{\Longrightarrow} N_{i+1}$ for $i = 0,1,\dots,k-1$ then there exists $\tau \in \overline{B_u}$ such that*

$$(f_i \circ f_{i-1} \circ \cdots \circ f_0)(b(\tau)) \in |N_{i+1}|, \quad \textit{for} \quad i = 0,1,2,\dots,k-2,$$
$$(f_{k-1} \circ f_{k-2} \circ \cdots \circ f_0)(b(\tau)) \in v(\overline{B_{s(N_k)}}).$$

Proof: Let Z be an h-set, such that $|Z| = \overline{B_u}$, $u(Z) = u$, $s(Z) = 0$ and the homeomorphism c_Z is the identity.

We define set h-set $\widetilde{N}$ as a double stretched set N_0 in stable direction, namely

$$|\widetilde{N}| = c_{N_0}^{-1}(\overline{B_u} \times \overline{B_{s(N_0)}}(0,2))$$
$$c_{\widetilde{N}} = V \circ c_{N_0}$$

where

$$V : \mathbb{R}^u \times \mathbb{R}^{s(N_0)} \ni (p,q) \to \left(p, \frac{1}{2}q\right) \in \mathbb{R}^u \times \mathbb{R}^{s(N_0)}. \tag{29}$$

Define $W : |Z| \to \mathbb{R}^{\dim(N_0)}$ by

$$W(z) = \begin{cases} b(2z) & \text{for } \|z\| \le \frac{1}{2} \\ c_{\widetilde{N}}^{-1}\left(2\|z\| \cdot c_{\widetilde{N}}\left(b\left(\frac{z}{\|z\|}\right)\right)\right) & \text{for } \|z\| \in \left[\frac{1}{2},1\right] \end{cases}$$

The map W is continuous since both formulas coincide for $\|z\| = \frac{1}{2}$. We will show that $Z \overset{W}{\Longrightarrow} \widetilde{N}$. The required homotopy $h : [0,1] \times \overline{B_u} \to \mathbb{R}^{\dim(\widetilde{N})}$ is given by

$$h(t,x) = \begin{cases} V(h_b(t,2x)) & \text{for } \|x\| \le \frac{1}{2} \\ 2\|x\| \cdot V\left(h_b\left(t,\frac{x}{\|x\|}\right)\right) & \text{for } \|x\| \in \left[\frac{1}{2},1\right] \end{cases}$$

where h_b is the homotopy from the definition of horizontal disc b in N_0. We will show that h satisfies (4–8). We have

$$W_c(x) = h(0,x) = \begin{cases} c_{\widetilde{N}}(b(2x)) & \text{for } \|x\| \leq \frac{1}{2} \\ 2\|x\| \cdot c_{\widetilde{N}}\left(b\left(\frac{x}{\|x\|}\right)\right) & \text{for } \|x\| \in \left[\frac{1}{2}, 1\right] \end{cases}$$

which proves that h satisfies (4). Let us fix $t \in [0,1]$ and $x \in \partial B_u$. Then $\|x\| = 1$, $(a,b) := h_b\left(t, \frac{x}{\|x\|}\right) \in \partial B_u \times \overline{B_{s(N_0)}}$ and consequently

$$h(t,x) = 2\|x\| \cdot V(a,b) = (2a, b) \notin \partial B_u \times \overline{B_{s(N_0)}}.$$

This shows that h satisfies (5).

Let us fix $t \in [0,1]$ and $x \in \overline{B_u}$. If $\|x\| \leq \frac{1}{2}$ then

$$h(t,x) = V(h_b(t,2x)) \in V\left(\overline{B_u} \times \overline{B_{s(\widetilde{N})}}\right) = \overline{B_u} \times \overline{B_{s(\widetilde{N})}}(0, \tfrac{1}{2}).$$

This shows that $h(t,x) \notin \widetilde{N}^+$ for $t \in [0,1]$ and $\|x\| \leq \frac{1}{2}$. If $\frac{1}{2} < \|x\| \leq 1$ then

$$h(t,x) = 2\|x\| \cdot V(a,b) = \|x\|(2a, b), \tag{30}$$

where $(a,b) := h_b\left(t, \frac{x}{\|x\|}\right)$. Since $\frac{x}{\|x\|} \in \partial B_u$, from (11) we conclude that $a \in \partial B_u$. From (30) we obtain $h(t,x) \notin \overline{B_u} \times \overline{B_s(\widetilde{N})}$. This shows that h satisfies (6) for $t \in [0,1]$ and $\frac{1}{2} < \|x\| \leq 1$.

Finally, let us observe that

$$h(1,x) = \begin{cases} V(2x, 0) & \text{for } \|x\| \leq \frac{1}{2} \\ 2\|x\| V\left(\frac{x}{\|x\|}, 0\right) & \text{for } \|x\| \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and from (29) $h(1,x) = (2x, 0)$, which shows that h satisfies (7–8). Thus $Z \overset{W}{\Longrightarrow} \widetilde{N}$.

Now we define $\bar{f}_0 : \widetilde{N} \to \mathbb{R}^{\dim(N_1)}$ such that $\bar{f}_0|_{|N_0|} = f_0$ and $\widetilde{N} \overset{\bar{f}_0}{\Longrightarrow} N_1$. Put

$$\bar{f}_0(x) = \begin{cases} f(x) & \text{for } x \in |N_0| \\ (f \circ c_{N_0}^{-1} \circ \pi \circ c_{N_0})(x) & \text{for } x \in |\widetilde{N}| \setminus |N_0| \end{cases}$$

where

$$\pi : \mathbb{R}^u \times \mathbb{R}^{s(N_1)} \ni (p,q) \to \left(p, \frac{q}{\|q\|}\right) \in \mathbb{R}^u \times \mathbb{R}^{s(N_1)}.$$

Let $\bar{h}$ be the homotopy from the covering relation $N_0 \overset{f_0}{\Longrightarrow} N_1$. One can easily verify that the homotopy

$$h(t,p,q) = \begin{cases} \bar{h}(t,p,2q) & \text{for } (p,q) \in \overline{B_u} \times \overline{B_{s(N_0)}}\left(0, \frac{1}{2}\right) \\ \bar{h}(t,p,1) & \text{for } x \in \overline{B_u} \times \left(\overline{B_{s(N_0)}} \setminus \overline{B_{s(N_0)}}\left(0, \frac{1}{2}\right)\right) \end{cases}$$

satisfies (4–8). Therefore $\widetilde{N} \overset{\bar{f}_0}{\Longrightarrow} N_1$. Define

$$\widetilde{f}_0 : |Z| \times |N_0| \ni (z,x) \to \bar{f}_0(x) \in \mathbb{R}^{\dim(N_1)},$$
$$\widetilde{f}_i : |Z| \times |N_i| \ni (z,x) \to f_i(x) \in \mathbb{R}^{\dim(N_{i+1})} \quad \text{for } i = 1, 2, \ldots, k-1.$$

From Theorem 3.3 applied to sequence

$$Z \overset{W}{\Longrightarrow} \widetilde{N} \overset{\widetilde{f}_0(z,\cdot)}{\Longrightarrow} N_1 \overset{\widetilde{f}_1(z,\cdot)}{\Longrightarrow} N_2 \overset{\widetilde{f}_2(z,\cdot)}{\Longrightarrow} \cdots N_{k-1} \overset{\widetilde{f}_{k-1}(z,\cdot)}{\Longrightarrow} N_k$$

we obtain that there exists $z \in |Z|$ such that

$$W(z) \in |\widetilde{N}|,$$
$$(\widetilde{f}_i(z,\cdot) \circ \widetilde{f}_{i-1}(z,\cdot) \circ \cdots \circ \widetilde{f}_1(z,\cdot) \circ \widetilde{f}_0(z,\cdot))(W(z)) \in |N_{i+1}|, \quad \text{for} \quad i = 0,1,2,\ldots,k-2,$$
$$(\widetilde{f}_{k-1}(z,\cdot) \circ \widetilde{f}_{k-2}(z,\cdot) \circ \cdots \circ \widetilde{f}_1(z,\cdot) \circ \widetilde{f}_0(z,\cdot))(W(z)) \in v(\overline{B_{s(N_k)}}).$$

From the definition of W we get that $W(z) \notin \widetilde{N}$ for $\frac{1}{2} < \|z\| \leq 1$. Hence, $W(z) = b(2\lambda)$ for some $\lambda \in \overline{B_u}(0, \frac{1}{2})$. Therefore $W(z) \in |N_0|$. Put $\tau = 2\lambda$. Since $\widetilde{f}_0(z,x) = f_0(x)$ for all $(z,x) \in |Z| \times |N_0|$ and $\widetilde{f}_i(z,\cdot) = f_i$ for all $z \in |Z|$ we obtain that

$$(f_i \circ f_{i-1} \circ \cdots \circ f_0)(b(\tau)) \in |N_i|, \quad \text{for} \quad i = 0,1,2,\ldots,k-2,$$
$$(f_{k-1} \circ f_{k-2} \circ \cdots \circ f_0)(b(\tau)) \in v(\overline{B_{s(N_k)}})$$

and the proof is finished. □

Corollary 3.10. *Let N_i, $i = 0,1,\ldots$ be h-sets such that $u(N_i) = u$ for $i = 0,1,\ldots$ and let $f_i : |N_i| \to \mathbb{R}^{\dim(N_{i+1})}$, $i = 0,1,\ldots$ be continuous. Let $b : \overline{B_u} \to |N_0|$ be a horizontal disc in N_0. If $N_i \overset{f_i}{\Longrightarrow} N_{i+1}$ for $i = 0,1,\ldots$, then there exists $\tau \in \overline{B_u}$ such that*

$$(f_i \circ f_{i-1} \circ \cdots \circ f_0)(b(\tau)) \in |N_{i+1}|, \quad \textit{for} \quad i = 0,1,2,\ldots$$

Proof: By Theorem 3.9, for every finite sequence

$$N_0 \overset{f_0}{\Longrightarrow} N_1 \overset{f_1}{\Longrightarrow} \cdots \overset{f_{k-1}}{\Longrightarrow} N_k$$

we can find $\tau_k \in \overline{B_u}$ such that

$$(f_{k-1} \circ \cdots \circ f_0)(b(\tau)) \in |N_k|$$

Since $\overline{B_u}$ is compact we can find an accumulation point τ of the sequence $(\tau_k)_{k\in\mathbb{N}}$. Obviously τ satisfies the assertion of the Corollary. □

4. Application to the Michelson system.

In this section we present an application of the method introduced in Section 3 to the Michelson system and we give the proofs of Theorems 1.1, 1.6, 1.2 and 1.3. The main theorems in this section, i.e. Theorems 4.3, 4.4 and 4.9 describe the geometry of heteroclinic and homoclinic solutions in the terms of admissible sequences for a suitable Poincaré map. Theorems 1.6, 1.2 and 1.3 are direct corollaries.

The proofs of the main theorems are based on several lemmas. Four of them were proved with computer assistance. The algorithms and ideas of computer assisted proofs will be presented in Section 5.

4.1. **Representation of h-sets.** In this section we deal with h-sets possessing exactly one unstable direction. Therefore we use the following representation. An h-set N in $\mathbb{R}^n$ may be defined by specifying a sequence $(x, u, s_1, \ldots, s_{n-1})$, where $x, u, s_i \in \mathbb{R}^n$, $i = 1,2,\ldots,n-1$ are such that $u, s_1, \ldots, s_{n-1}$ are linearly independent. We then set

$$\begin{aligned} |N| &:= \{v \in \mathbb{R}^n \mid \exists_{t_1,t_2,\ldots,t_n \in [-1,1]} \quad v = x + t_1 s_1 + \cdots + t_{n-1}s_{n-1} + t_n u\} \\ &= x + [-1,1]\cdot u + [-1,1]\cdot s_1 + \cdots + [-1,1]\cdot s_{n-1}. \end{aligned}$$

and take u as the nominally unstable direction and s_i as the nominally stable directions. The homeomorphism c_N is taken as the affine map $c_N(v) = M^{-1}(v-x)$, where $M = [u^T, s_1^T, \ldots, s_{n-1}^T]$ is a square matrix. In this representation $N_c = \overline{B_1} \times \overline{B_{n-1}} = [-1,1]^n$ is a product of unit balls in the maximum norm.

In such a situation we will write $N = \mathfrak{h}(x, u, s_1, s_2, \ldots, s_{n-1})$.

4.2. **Symbolic dynamics in the Michelson system.** Let $\Theta := \{(x, y, 0) \mid x, y, \in \mathbb{R}\}$ be a Poincaré section of the Michelson system. Since the third coordinate is equal to zero we will use (x, y) coordinates to represent the points in Θ. We define eleven h-sets $N_i = \mathfrak{h}(x_i, u_i, s_i)$, $i = 1, \ldots, 11$, where

$$\begin{aligned}
x_1 &= (0.0, 1.40), & u_1 &= (0.072, 0.04), & s_1 &= (-0.072, 0.04),\\
x_2 &= (0.0, 0.225), & u_2 &= (0.15, 0.1), & s_2 &= (-0.15, 0.1),\\
x_3 &= (1.218, 0.85), & u_3 &= (0.035, 0.049), & s_3 &= (-0.12, 0.06),\\
x_4 &= (0., -2.008), & u_4 &= (0.06, 0.1), & s_4 &= (-0.06, 0.1),\\
x_5 &= (-1.218, 0.85), & u_5 &= (0.12, 0.06), & s_5 &= (-0.035, 0.049),\\
x_6 &= (-1.4753, 0.715565), & u_6 &= (0.051, 0.03145), & s_6 &= (-0.03, 0.05),\\
x_7 &= (-0.889054, -0.182129), & u_7 &= (0.1122, 0.0627), & s_7 &= (-0.021, 0.0525),\\
x_8 &= (0.86, 1.0368), & u_8 &= (0.035, 0.049), & s_8 &= (-0.07, 0.03),\\
x_9 &= (1.4753, 0.715565), & u_9 &= (0.03, 0.05), & s_9 &= (-0.051, 0.03145),\\
x_{10} &= (0.889054, -0.182129), & u_{10} &= (0.021, 0.0525), & s_{10} &= (-0.1122, 0.0627),\\
x_{11} &= (-0.86, 1.0368), & u_{11} &= (0.07, 0.03), & s_{11} &= (-0.035, 0.049).
\end{aligned} \tag{31}$$

These sets are presented in Fig. 8.

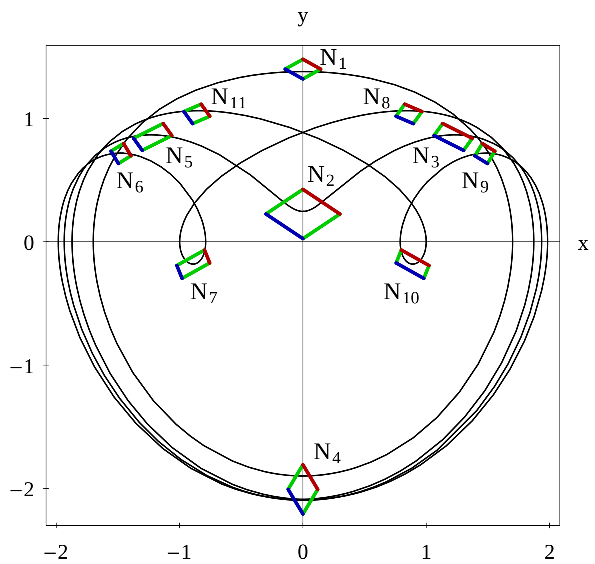

FIGURE 8. The sets N_i, $i = 1, \ldots, 11$. These sets have been chosen as neighborhoods of the intersection of periodic orbits with the Poincaré section Θ.

Let $N = \bigcup_{i=1}^{11} |N_i|$. Let $P_c : \Theta \multimap \Theta$ denote the Poincaré map for the Michelson system (1) with parameter c. We define $P : [c_{\min}, c_{\max}] \times \Theta \multimap \Theta$ by

$$\begin{aligned}
\operatorname{dom}(P) &:= \{(c, x) \mid c \in [c_{\min}, c_{\max}], x \in \operatorname{dom}(P_c)\},\\
P(c, x) &:= P_c(x) \quad \text{for } (c, x) \in \operatorname{dom}(P).
\end{aligned}$$

Lemma 4.1. *The map P is well defined and continuous on $[c_{\min}, c_{\max}] \times N$. Moreover, for each parameter value $c \in [c_{\min}, c_{\max}]$ the following covering relations hold*

$$
\begin{aligned}
N_4 \overset{P(c,\cdot)}{\Longrightarrow} N_1 \overset{P(c,\cdot)}{\Longrightarrow} N_4,\\
N_4 \overset{P(c,\cdot)}{\Longrightarrow} N_5 \overset{P(c,\cdot)}{\Longrightarrow} N_2 \overset{P(c,\cdot)}{\Longrightarrow} N_3 \overset{P(c,\cdot)}{\Longrightarrow} N_4,\\
N_4 \overset{P(c,\cdot)}{\Longrightarrow} N_6 \overset{P(c,\cdot)}{\Longrightarrow} N_7 \overset{P(c,\cdot)}{\Longrightarrow} N_8 \overset{P(c,\cdot)}{\Longrightarrow} N_4,\\
N_4 \overset{P(c,\cdot)}{\Longrightarrow} N_{11} \overset{P(c,\cdot)}{\Longrightarrow} N_{10} \overset{P(c,\cdot)}{\Longrightarrow} N_9 \overset{P(c,\cdot)}{\Longrightarrow} N_4.
\end{aligned}
\tag{32}
$$

A computer assisted proof of the above lemma will be given in Section 5. The numerical evidence of the existence of covering relations (32) is presented in Fig. 9. It is important to note that there are four loops of covering relations corresponding to four periodic orbits as it is presented in Fig. 8. Moreover, these four loops contain N_4 as a common set. This allows us to construct essentially different chains of covering relations of an arbitrary length.

Proof of Theorem 1.1. The proof follows directly from Lemma 4.1 and Theorem 2.5. Let us fix $c \in [c_{\min}, c_{\max}]$ and put $\Sigma_4 := \{1, 5, 6, 11\}^{\mathbb{Z}}$ and define the shift map

$$\sigma : \Sigma_4 \ni (x_i)_{i\in\mathbb{Z}} \longrightarrow (x_{i+1})_{i\in\mathbb{Z}} \in \Sigma_4.$$

Let $\pi : |N_1| \cup |N_5| \cup |N_6| \cup |N_{11}| \multimap \Sigma_4$ be defined as follows:

$$
\begin{aligned}
\operatorname{dom}(\pi) &:= \left\{x \in \Theta \mid \forall j \in \mathbb{Z},\ x \in \operatorname{dom}(P_c^{4j}), P_c^{4j}(x) \in |N_1| \cup |N_5| \cup |N_6| \cup |N_{11}|\right\},\\
\pi(x) = (i_j)_{j\in\mathbb{Z}} &\Longleftrightarrow P_c^{4j}(x) \in |N_{i_j}| \quad \text{for } x \in \operatorname{dom}(\pi).
\end{aligned}
$$

Since $|N_i| \cap |N_j| = \emptyset$, for $i \neq j$ we get that π is well defined. Moreover, for $x \in \operatorname{dom}(\pi)$ we have $\pi(P_c^4(x)) = \sigma(\pi(x))$. From Theorem 2.5 and (32) it follows that the preimage of any sequence of symbols in Σ_4 is nonempty and if $s \in \Sigma_4$ is periodic then $\pi^{-1}(s)$ contains a periodic point of P_c^4. Thus the fourth iteration of the Poincaré map restricted to $\operatorname{dom}(\pi)$ is semiconjugate to the full shift on four symbols. □

4.3. **The existence of heteroclinic solutions connecting x_- with x_+.** Observe that the interval of parameter values $[c_{\min}, c_{\max}]$ can be represented as a one dimensional h-set with the unstable direction only

$$C = \mathfrak{h}((c_{\min} + c_{\max})/2, (c_{\max} - c_{\min})/2).$$

For each parameter value $c \in |C| = [c_{\min}, c_{\max}]$ there exists a pair of equilibrium points $(\pm\sqrt{2}c, 0, 0)$. The equilibrium point $(-\sqrt{2}c, 0, 0)$ is hyperbolic with one dimensional unstable manifold. One branch of this manifold leaves the equilibrium point and escapes quickly to infinity (see branch E in Fig. 10 left panel). The second branch can intersect the Poincaré section. Let $W : |C| \to \Theta$ be the map such that for $c \in |C|$ $W(c)$ denotes the second intersection of this branch with the Poincaré section Θ.

Lemma 4.2. *The function $W : |C| \to \Theta$ is well defined and continuous. Moreover, $C \overset{W}{\Longrightarrow} N_4$.*

A computer assisted proof of the above lemma will be discussed in Section 5.2. The numerical evidence of the existence of the covering relation $C \overset{W}{\Longrightarrow} N_4$ is presented in Fig. 10.

Now we present the main theorem in this section.

Theorem 4.3. *Let $(i_0, i_1, \ldots, i_k)$, $k \geq 0$ be an admissible sequence with respect to P_c, $c \in [c_{\min}, c_{\max}]$ such that $i_0 = 4$ and $i_k \in \{1, 2, 4\}$. Then there exist $c_0 \in [c_{\min}, c_{\max}]$ and a solution $u : \mathbb{R} \to \mathbb{R}^3$ of (1) with parameter value c_0 such that the following conditions hold true*

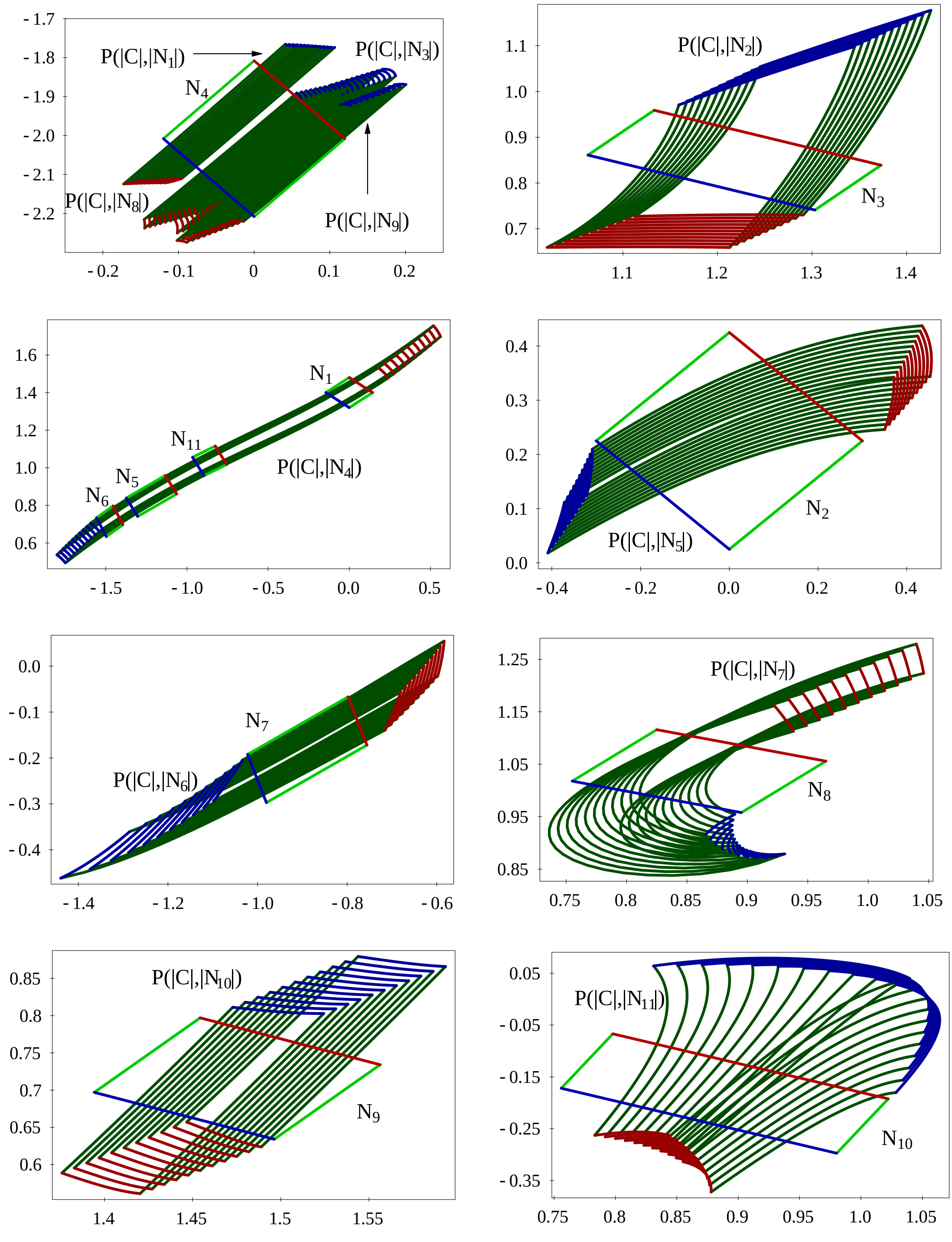

FIGURE 9. The sets N_i, $i = 1, \ldots, 11$ and their images $P(|C|, N)$. The numerical evidence of covering relations established in Lemma 4.1. When in color: red and blue colors correspond to N^- and its image, green color corresponds to N^+ and its image.

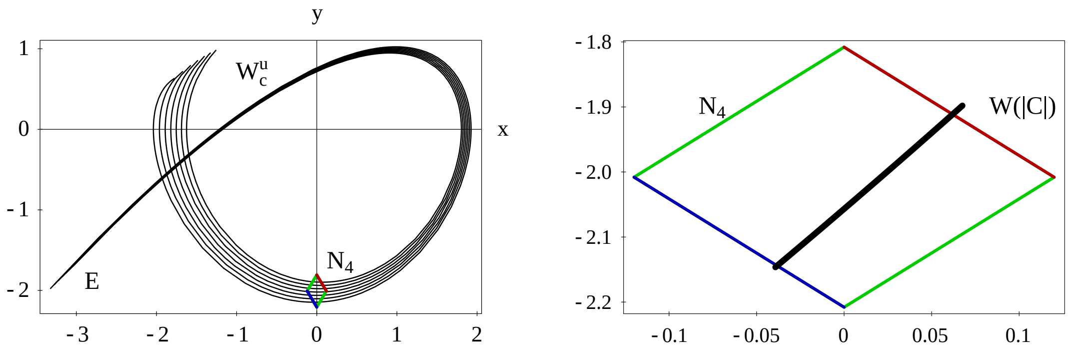

FIGURE 10. (left) A part of the unstable manifold of $(-\sqrt{2}c, 0, 0)$ projected onto (x, y) plane for certain parameter values $c \in [c_{\min}, c_{\max}]$, (right) the numerical evidence of the covering relation $C \overset{W}{\Longrightarrow} N_4$.

(1) $u(0) = (0, y, 0)$ *for some* $y \in \mathbb{R}$,
(2) *there exist real numbers* $0 = t_0 < t_1 < \ldots < t_k$ *such that* $u(t_j) \in R(|N_{i_{k-j}}|)$ *and* $u(-t_j) \in |N_{i_{k-j}}|$ *for* $j = 0, \ldots, k$,
(3) $\lim_{t\to\infty} u(t) = (\sqrt{2}c_0, 0, 0)$, $\lim_{t\to-\infty} u(t) = (-\sqrt{2}c_0, 0, 0)$.

Hence u *is an* R*-symmetric heteroclinic solution.*

Proof: Recall that the system (1) possesses the reversing symmetry R given by (2). Let

$$\mathrm{Fix}(R) = \{u \in \mathbb{R}^3 \mid R(u) = u\} = \{(0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\}.$$

Define the vertical disc in N_{i_k}

$$b : \overline{B_1} \ni y \to x_{i_k} + y \cdot s_{i_k} + y \cdot u_{i_k} \in |N_{i_k}|,$$

where $x_{i_k}, u_{i_k}, s_{i_k}$ are defined in (31). The homotopy required in Definition 3.2 is given by

$$h_v : [0, 1] \times \overline{B_1} \ni (t, y) \to ((1-t)y, y) \in (N_{i_k})_c.$$

Observe that the unstable and stable vectors used to define h-sets N_1, N_2 and N_4 are symmetric, i.e. $R(s_i) = u_i$ for $i = 1, 2, 4$ and R is the reflection with respect to y axis. Moreover, $x_i \in \mathrm{Fix}(R)$ for $i = 1, 2, 4$. Therefore $b(\overline{B_1}) \subset \mathrm{Fix}(R)$. Now, Theorem 3.3 applied to the sequence of covering relations

$$C \overset{W}{\Longrightarrow} N_{i_0} = N_4 \overset{P(c,\cdot)}{\Longrightarrow} N_{i_1} \overset{P(c,\cdot)}{\Longrightarrow} \cdots \overset{P(c,\cdot)}{\Longrightarrow} N_{i_k}$$

and the vertical disc b in N_{i_k} implies that there exists $c_0 \in |C| = [c_{\min}, c_{\max}]$ such that $W(c_0) \in |N_{i_0}|$, $P_{c_0}^j(W(c_0)) \in |N_{i_j}|$, for $j = 0, \ldots, k$ and $P_{c_0}^k(W(c_0)) \in b(\overline{B_1}) \subset \mathrm{Fix}(R)$. Put $u_0 := P_{c_0}^k(W(c_0))$. Let $u : \mathbb{R} \multimap \mathbb{R}^3$ be the solution of (1) with $c = c_0$ and the initial condition $u(0) = u_0$.

From the definition of the Poincaré map it follows that there exists a sequence of real numbers $0 = t_0 < t_1 < \ldots < t_k$ such that $u(-t_j) = P_{c_0}^{-j}(u_0) \in |N_{i_{k-j}}|$, for $j = 0, \ldots, k$. From the definition of W and from $u(-t_k) = P_{c_0}^{-k}(u_0) = W(c_0)$ it follows that u is defined for all $t < 0$ and $\lim_{t\to-\infty} u(t) = (-c_0\sqrt{2}, 0, 0)$.

Since $u(0) = u_0 \in \mathrm{Fix}(R)$, the symmetry property of (1) implies that u is defined for all $t \in \mathbb{R}$, $u(t_j) = R(u(-t_j)) \in R(|N_{i_{k-j}}|)$ and $\lim_{t\to\infty} u(t) = R(-c_0\sqrt{2}, 0, 0) = (c_0\sqrt{2}, 0, 0)$. $\square$

Proof of Theorem 1.2. The assertion is a direct consequence of Theorem 4.3, because we can find countable set of sequences $(i_0, \ldots, i_k)$ satisfying the assumptions of Theorem 4.3, for example $(4, 1, \underbrace{4, 5, 2, 3, 4, 5, 2, 3, \ldots, 4, 5, 2, 3}_{n \text{ times}}, 4, 1) \in \mathbb{N}^{4n+4}$, $n > 0$. Clearly, each sequence gives a geometrically different heteroclinic solution. □

The heteroclinic solutions presented in Fig. 2 correspond to admissible sequences $(4, 5, 2)$, $(4, 5, 2, 3, 4, 1)$ and $(4, 11, 10, 9, 4)$ respectively. The analytic form (3) of the heteroclinic solution found by Kuramoto and Tsuzuki [12] is the simplest one of the family of heteroclinic solutions resulting from Theorem 4.3 and it corresponds to the shortest admissible sequence $i_0 = 4$.

4.4. **The existence of Shilnikov homoclinic orbits.** The goal of this section is to prove the following theorem.

Theorem 4.4. *Let $(i_0, i_1, \ldots, i_k)$, $k > 0$ be an admissible sequence with respect to P_c, $c \in [c_{\min}, c_{\max}]$ such that $i_0 = i_k = 4$. Then there exists $c_0 \in [c_{\min}, c_{\max}]$ and the solution $u : \mathbb{R} \to \mathbb{R}^3$ of (1) with parameter $c = c_0$ such that the following conditions hold true*

- *there exists a sequence of real numbers $0 = t_0 < t_1 < \ldots < t_k$ such that $u(t_j) \in |N_{i_j}|$*
- $\lim_{t\to\pm\infty} u(t) = (-c_0\sqrt{2}, 0, 0)$.

Let $\phi : [c_{\min}, c_{\max}] \times \mathbb{R} \times \mathbb{R}^3 \nrightarrow \mathbb{R}^3$ denote the local dynamical system induced by (1) for parameter values $c \in [c_{\min}, c_{\max}]$. Let $x_-(c) = (-c\sqrt{2}, 0, 0)$ be the equilibrium point for $\phi(c, \cdot, \cdot)$. The proof of Theorem 4.4 consists of the following steps

(1) we define a three dimensional h-set H centered at the equilibrium point $x_-(c_{\text{mid}})$ for a fixed $c_{\text{mid}} \in [c_{\min}, c_{\max}]$ and we find a time T_H such that $H \overset{\phi(c,T_H,\cdot)}{\Longrightarrow} H$ for $c \in [c_{\min}, c_{\max}]$.
(2) we define an h-set M as a subset of N_4 and we find a time T_M such that $M \overset{\phi(c,T_M,\cdot)}{\Longrightarrow} H$ for $c \in [c_{\min}, c_{\max}]$
(3) finally, we apply Corollary 3.4 to the sequence of covering relations

$$C \overset{W}{\Longrightarrow} N_{i_0} \overset{P(c,\cdot)}{\Longrightarrow} \cdots \overset{P(c,\cdot)}{\Longrightarrow} N_{i_{k-1}} \overset{P(c,\cdot)}{\Longrightarrow} M \overset{\phi(c,T_M,\cdot)}{\Longrightarrow} H \overset{\phi(c,T_H,\cdot)}{\Longrightarrow} H \overset{\phi(c,T_H,\cdot)}{\Longrightarrow} \cdots \tag{33}$$

in order to prove the existence of homoclinic solution. We use a certain energy function to prove that the trajectory corresponding to the sequence (33) is forward asymptotic to the equilibrium point.

Observe that the linearized flow $\phi(c, \cdot, \cdot)$ in $x_-(c)$ possesses one real eigenvalue $\lambda_1(c) > 0$ and a pair of complex eigenvalues $\lambda_2(c), \lambda_3(c)$ with negative real parts. Therefore, we have a one dimensional unstable manifold and a two dimensional stable manifold. We define a three-dimensional h-set $H = \mathfrak{h}(x, u, s_1, s_2)$ (see Section 4.1), where $x = x_-(c_{\text{mid}}) = x_-(0.5(c_{\min}+c_{\max}))$ and the vectors

$$\begin{aligned}
u &= (1.7385347229815995, 1.3185350670276454, 1),\\
s_1 &= (-0.51893229665560336, -0.24073732860908956, 1),\\
s_2 &= (0.36569474342908287, -0.75953061692169033, 0)
\end{aligned}$$

are good numerical approximations of the stable and unstable eigenvectors in $x_-(c)$. In fact the stable and unstable eigenvectors may be computed exactly, but it is not necessary for our method (see (49–50)).

Lemma 4.5. *Let $T_H := 1.4$ and $\Phi_H := \phi(\cdot, T_H, \cdot)$. Then for all $u \in |H|$ and all $c \in [c_{\min}, c_{\max}]$ the solution of (1) with the initial condition u is defined on the interval $[0, T_H]$. Moreover, for all $c \in [c_{\min}, c_{\max}]$*

$$\phi(c, [0, T_H], |H|) \subset \{(x, y, z) \in \mathbb{R}^3 \mid x < 0\} \quad \text{and} \quad H \overset{\Phi_H(c,\cdot)}{\Longrightarrow} H.$$

A computer assisted proof of the above lemma will be discussed in Section 5.1.

Recall that $N_4 = \mathfrak{h}(x_4, u_4, s_4)$, where x_4, u_4, s_4 are defined by (31). The numerical simulation shows that the intersection of the stable manifold of $x_-(c)$ with the Poincaré section Θ crosses the set N_4. Therefore, we define the set M as a subset of N_4 containing a part of this intersection. Put

$$M = \mathfrak{h}(x_4 - 0.7u_4, 0.3u_4, s_4). \tag{34}$$

The set M is presented in Fig. 11.

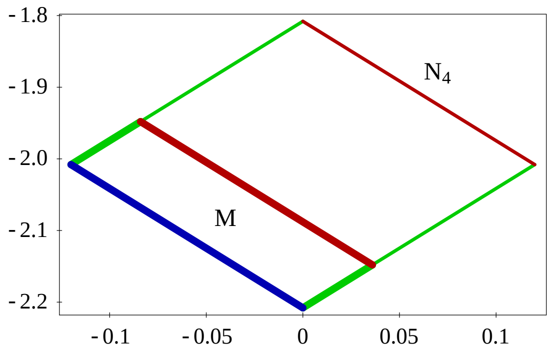

FIGURE 11. The h-set M.

Lemma 4.6. *For all $c \in [c_{\min}, c_{\max}]$ the following covering relations hold*

$$N_1 \overset{P(c,\cdot)}{\Longrightarrow} M, \quad N_3 \overset{P(c,\cdot)}{\Longrightarrow} M, \quad N_8 \overset{P(c,\cdot)}{\Longrightarrow} M, \quad N_9 \overset{P(c,\cdot)}{\Longrightarrow} M.$$

Proof: From Lemma 4.1 we know that

$$N_1 \overset{P(c,\cdot)}{\Longrightarrow} N_4, \quad N_3 \overset{P(c,\cdot)}{\Longrightarrow} N_4, \quad N_8 \overset{P(c,\cdot)}{\Longrightarrow} N_4, \quad N_9 \overset{P(c,\cdot)}{\Longrightarrow} N_4.$$

Let us fix $i \in \{1, 3, 8, 9\}$ and $c \in [c_{\min}, c_{\max}]$. We will show that $N_i \overset{P(c,\cdot)}{\Longrightarrow} M$. Let $h_1 : [0,1] \times N_{i,c} \to \mathbb{R}^2$ and $A : \mathbb{R} \to \mathbb{R}$ be a homotopy and a linear map from the definition of the covering relation $N_i \overset{P(c,\cdot)}{\Longrightarrow} N_4$. Notice, that $\left(c_M \circ c_{N_4}^{-1}\right)(a, b) = \left(\frac{7+10a}{3}, b\right)$. Define $h_2 : [0,1] \times N_{i,c} \to \mathbb{R}^2$ by

$$h_2(t, p, q) = t\left(\frac{10}{3}A(p), 0\right) + (1-t)\left(c_M \circ c_{N_4}^{-1}\right)(A(p), 0) = \frac{1}{3}\left(10A(p) + 7(1-t), 0\right)$$

We define the required homotopy $h : [0,1] \times N_{i,c} \to \mathbb{R}^2$ as

$$h(t, x) = \begin{cases} (c_M^{-1} \circ c_N \circ h_1)(2t, x) & \text{for } t \in \left[0, \frac{1}{2}\right] \\ h_2(2t-1, x) & \text{for } t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Since $h_1(1, p, q) = h_2(0, p, q) = \left(c_M \circ c_{N_4}^{-1}\right)(A(p), 0)$ we get that h is continuous. We need to prove that h satisfies conditions (4–8).

Condition (4) holds for h just from its definition, i.e.

$$h(0,\cdot) = h_1(0,\cdot) = c_M \circ c_{N_4}^{-1} \circ c_{N_4} \circ P(c,\cdot) \circ c_{N_i}^{-1} = c_M \circ P(c,\cdot) \circ c_{N_i}^{-1}$$

Let us fix $t \in \left[0, \frac{1}{2}\right]$. We know that h_1 satisfies condition (5), hence for $(p,q) \in N_{i,c}^-$ the maximum norm $\|h(t,p,q)\|_1 > 1$. It is obvious that if $\|(a,b)\|_1 > 1$ then $\|\left(\frac{7+10a}{3}, b\right)\|_1 > 1$ too. Therefore h satisfies (5) for $t \in \left[0, \frac{1}{2}\right]$.

If $t \in \left[\frac{1}{2}, 0\right]$ then $h(t,p,q) = h_2(2t-1,p,q) = \frac{1}{3}\left(10A(p) + 14(1-t), 0\right)$. From (8) we know that $|A(p)| > 1$ and consequently $|\frac{1}{3}\left(10A(p) + 14(1-t)\right)| > 1$ for $t \in \left[\frac{1}{2}, 1\right]$. It shows that h satisfies (5).

Condition (6) is obvious for $t \in \left[\frac{1}{2}, 1\right]$, because h_2 is zero on the last coordinate. Let us fix $t \in \left[0, \frac{1}{2}\right]$. We know that h_1 satisfies (6), i.e. $(a,b) := h_1(2t,p,q) \notin [-1,1] \times \{-1,1\}$ for $t \in \left[0, \frac{1}{2}\right]$. Hence,

$$h(t,p,q) = (c_M \circ c_{N_4}^{-1} \circ h_1)(2t,p,q) = \left(\frac{10a+7}{3}, b\right) \notin [-1,1] \times \{-1,1\} = M_c^+.$$

This shows that h satisfies (6).

Finally, let us observe that $h(1,p,q) = \left(\frac{10}{3}A(p), 0\right)$ and $\frac{10}{3}A$ is a linear map which satisfies (7–8) as well as A. The proof is completed. □

Lemma 4.7. *Let $T_M := 6$ and put $\Phi_M := \phi(\cdot, T_M, \cdot)$. Then ϕ is well defined on $[c_{\min}, c_{\max}] \times [0, T_M] \times |M|$. Moreover, $M \overset{\Phi_M(c,\cdot)}{\Longrightarrow} H$ for all $c \in [c_{\min}, c_{\max}]$.*

A computer assisted proof of the above lemma will be discussed in Section 5.1.

Lemma 4.8. *Let us fix $c \in [c_{\min}, c_{\max}]$. Assume $u \in |H|$ is such that*

$$\left(\Phi_H^n(c,\cdot)\right)(u) \in |H|, \qquad \text{for } n > 0. \tag{35}$$

Then $\lim_{t\to\infty} \phi(c,t,u) = (-c\sqrt{2}, 0, 0)$.

Proof: Let

$$V(x,y,z) = z^2/2 + y(y - 2c^2 + x^2)/2 \tag{36}$$

be an energy function. Its derivative along the solution is equal to

$$\frac{d}{dt}V(x(t),y(t),z(t)) = x(t)(y(t))^2. \tag{37}$$

From Lemma 4.5 we have

$$\phi(c,[0,T_H],|H|) \subset \{(x,y,z) \in \mathbb{R}^3 \mid x < 0\}. \tag{38}$$

From (35) we know that the trajectory of u is defined on interval $[0,\infty)$ and from (38) we obtain $\phi(c,[0,\infty),u) \subset \Phi_H(c,[0,T_H],|H|)$. Since $\Phi_H(c,[0,T_H],|H|)$ is a compact set, we get that the ω-limit set $\omega(u)$ is a nonempty compact set and it satisfies

$$\omega(u) \subset \{(x,y,z) \in \mathbb{R}^3 \mid x < 0\}. \tag{39}$$

From (37–39) we obtain that the x coordinate is nonzero on $\omega(u)$ and y coordinate is equal to zero on $\omega(u)$. Hence $x' = y = 0$ and x coordinate must be constant on $\omega(u)$. Therefore,

$$z' = c^2 - y - x^2/2$$

is a constant function on $\omega(u)$. Since $\omega(u)$ is bounded we get $z' = 0$ on $\omega(u)$. Thus $\omega(u)$ is a single point and therefore it must be equal to $\{(-c\sqrt{2}, 0, 0)\}$, i.e. the unique equilibrium point in $\phi(c,[0,T_H],|H|)$. □

Now we are in the position to present the proof of Theorem 4.4.

Proof of Theorem 4.4. Let $(i_0, i_1, \ldots, i_k)$, $k > 0$ be an admissible sequence with respect to P, such that $i_0 = i_k = 4$. We have

$$N_4 = N_{i_0} \overset{P(c,\cdot)}{\Longrightarrow} N_{i_1} \overset{P(c,\cdot)}{\Longrightarrow} \cdots \overset{P(c,\cdot)}{\Longrightarrow} N_{i_k} = N_4.$$

From Lemma 4.1 it follows that $i_{k-1} \in \{1, 3, 8, 9\}$. From this and Lemma 4.6 we get

$$N_{i_0} \overset{P(c,\cdot)}{\Longrightarrow} N_{i_1} \overset{P(c,\cdot)}{\Longrightarrow} \cdots \overset{P(c,\cdot)}{\Longrightarrow} N_{i_{k-1}} \overset{P}{\Longrightarrow} M.$$

Now, Lemma 4.2, Lemma 4.5 and Lemma 4.7 imply that there exists the following sequence of covering relations

$$C \overset{W}{\Longrightarrow} N_{i_0} \overset{P(c,\cdot)}{\Longrightarrow} N_{i_1} \overset{P(c,\cdot)}{\Longrightarrow} \cdots \overset{P(c,\cdot)}{\Longrightarrow} N_{i_{k-1}} \overset{P(c,\cdot)}{\Longrightarrow} M \overset{\Phi_M(c,\cdot)}{\Longrightarrow} H \overset{\Phi_H(c,\cdot)}{\Longrightarrow} H \overset{\Phi_H(c,\cdot)}{\Longrightarrow} \cdots. \tag{40}$$

From Corollary 3.4 we get that there exists $c_0 \in [c_{\min}, c_{\max}] = |C|$, such that

$$W(c_0) \in |N_{i_0}|, \tag{41}$$

$$P^j(c_0, \cdot)(W(c_0)) \in |N_{i_j}|, \quad \text{for } j = 1, \ldots, k, \tag{42}$$

$$\left(\Phi_H^n(c_0, \cdot) \circ \Phi_M(c_0, \cdot) \circ P^k(c_0, \cdot)\right)(W(c_0)) \in |H|, \quad \text{for } n = 1, 2, \ldots \tag{43}$$

Put $u_0 = W(c_0) \in |N_{i_0}|$ and define $u(t) = \phi(c_0, t, u_0)$. From (42) and the definition of the Poincaré map there are numbers $0 = t_0 < t_1 < \cdots < t_k$ such that

$$u(t_j) \in |N_{i_j}|, \quad j = 0, \ldots, k.$$

From (43) and Lemma 4.8 we obtain that u is defined for $t > 0$ and

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} \phi(c_0, t, \Phi_M(c_0, u(t_k))) = (-c_0\sqrt{2}, 0, 0).$$

From (41) we know that u_0 lies in the unstable manifold of $x_-(c_0)$. Hence, $u(t)$ is defined for all $t < 0$ and

$$\lim_{t\to-\infty} u(t) = (-c_0\sqrt{2}, 0, 0).$$

□

The assertion of Theorem 1.6 is a straightforward corollary from Theorem 4.4.

Proof of Theorem 1.6. $\{(4, 1, 4, \ldots, 1, 4) \in \mathbb{N}^{2n+1}, n > 0\}$ is a countable set of admissible sequences satisfying the assumption of Theorem 4.4 and providing geometrically different Shilnikov homoclinic solutions. □

4.5. **The existence of heteroclinic chains between equilibrium points.** It is proved in [24, Thm.3.1] that the Michelson system possesses a countable family of symmetric heteroclinic connections between x_+ and x_- for the parameter value $c = 1$. In this section we prove a similar result for all parameter values $c \in [c_{\min}, c_{\max}]$. The main reason for which we present this result is that it implies together with Theorem 4.3 that for a countable set of parameter values there exist infinitely many heteroclinic loops between equilibrium points. It is proved in [13] that if there exists a heteroclinic loop between equilibrium points in a reversible system then a countable infinity of heteroclinic and periodic orbits close to such cycle exists.

The following theorem is the main result of this section.

Theorem 4.9. *Let us fix $c \in [c_{\min}, c_{\max}]$ and let $(i_0, i_1, \ldots, i_k)$, $k \geq 0$ be an admissible sequence with respect to P_c such that $i_0 \in \{1, 2, 4\}$, $i_k = 4$. Then there exists a solution u of the Michelson system (1) with parameter value c satisfying the following properties:*

(1) *the solution u is defined for all $t \in \mathbb{R}$,*
(2) *there exists a sequence $0 = t_0 < t_1 < \ldots < t_k$ such that $u(t_j) \in |N_{i_j}|$, $u(-t_j) \in R(|N_{i_j}|)$ for $j = 1, \ldots, k$ and $u(t_0) \in |N_{i_0}| \cap \mathrm{Fix}(R)$,*
(3) $\lim_{t\to\infty} u(t) = (-\sqrt{2}c, 0, 0)$ *and* $\lim_{t\to-\infty} u(t) = (\sqrt{2}c, 0, 0)$.

Proof: Let us fix $c \in [c_{\min}, c_{\max}]$ and consider the case $k > 0$. Since $i_k = 4$, from Lemma 4.1 we know that $i_{k-1} \in \{1, 3, 8, 9\}$. From Lemmas 4.5, 4.6 and 4.7 we obtain that there exists the following chain of covering relations

$$N_{i_0} \overset{P(c,\cdot)}{\Longrightarrow} N_{i_1} \overset{P(c,\cdot)}{\Longrightarrow} \cdots N_{i_{k-1}} \overset{P(c,\cdot)}{\Longrightarrow} M \overset{\Phi_M(c,\cdot)}{\Longrightarrow} H \overset{\Phi_H(c,\cdot)}{\Longrightarrow} H \overset{\Phi_H(c,\cdot)}{\Longrightarrow} \cdots \tag{44}$$

Define a horizontal disc in N_{i_0}

$$b : \overline{B_1} \ni x \to x_{i_0} + x \cdot s_{i_0} + x \cdot u_{i_0} \in |N_{i_0}|,$$

where $x_{i_0}, u_{i_0}, s_{i_0}$ are defined by (31). The homotopy required in the Definition 3.1 of horizontal disc is given by

$$h : [0,1] \times \overline{B_1} \ni (t, x) \to (x, (1-t)x) \in (N_{i_0})_c.$$

From Corollary 3.10 applied to the sequence (44) and the horizontal disc b in N_{i_0} we obtain that there exists $\tau \in \overline{B_1}$ such that

$$P^j(c, b(\tau)) \in |N_{i_j}|, \quad \text{fot} \quad j = 0, 1, \ldots, k \tag{45}$$
$$(\Phi_H^n(c,\cdot) \circ \Phi_M(c,\cdot) \circ P^k(c,\cdot))(b(\tau)) \in |H|, \quad \text{for} \quad n = 0, 1, \ldots \tag{46}$$

(notice that $|M| \subset |N_{i_k}| = |N_4|$). Put $u_0 = b(\tau)$ and let $u : \mathbb{R} \to \mathbb{R}^3$ be a solution of (1) with the parameter value c and the initial condition $u(0) = u_0$. From (46) we obtain that the solution u is defined for $t > 0$. From the definition of the Poincaré map and (45) there are real numbers $0 = t_0 < t_1 < \ldots < t_k$ such that $u(t_j) \in |N_{i_j}|$ for $j = 0, 1, \ldots, k$. From Lemma 4.8 and (46) we obtain $\lim_{t\to\infty} u(t) = (-\sqrt{2}c, 0, 0)$.

Since $b(\overline{B_1}) \subset \mathrm{Fix}(R)$, we get $u(0) = b(\tau) \in \mathrm{Fix}(R)$. Now, the symmetry argument implies that u is defined for $t < 0$ and

$$u(-t_j) \in R(|N_{i_j}|), \quad \text{for} \quad j = 0, 1, \ldots, k,$$
$$\lim_{t\to-\infty} u(t) = R((-\sqrt{2}c, 0, 0)) = (\sqrt{2}c, 0, 0).$$

If $k = 0$ then $i_0 = 4$. We can find a horizontal disc b in M such that $b(\overline{B_1}) \subset \mathrm{Fix}(R)$. Next we can repeat the above argument with the sequence

$$M \overset{\Phi_M(c,\cdot)}{\Longrightarrow} H \overset{\Phi_H(c,\cdot)}{\Longrightarrow} H \overset{\Phi_H(c,\cdot)}{\Longrightarrow} \cdots$$

instead of (44). This completes the proof. □

Now we give the proof of Theorem 1.3.

Proof of Theorem 1.3: We can find infinitely many admissible sequences satisfying assumptions of Theorem 4.9 providing geometrically different heteroclinic solutions. Take for example $(4, 1, 4, \underbrace{5, 2, 3, 4}_{n \text{ times}}, 1, 4) \in \mathbb{N}^{4n+5}$. □

5. Verifying covering relations with computer assistance.

In this section we explain how we can use the computer to verify the existence of covering relations. We have to prove the following assertions

- the existence of covering relations for the Poincaré map on sets N_i, $i = 1, \ldots, 11$ in Lemma 4.1
- the existence of covering relation $H \overset{\Phi_H(c,\cdot)}{\Longrightarrow} H$ and some inclusion in Lemma 4.5
- the existence of covering relation $M \overset{\Phi_M(c,\cdot)}{\Longrightarrow} H$ in Lemma 4.7
- the existence of covering relation $C \overset{W}{\Longrightarrow} N_4$ in Lemma 4.2

The last assertion requires a more sophisticated analysis and will be presented in the second part of this section.

5.1. General algorithms.

The algorithms presented here constitute a simple modification of the algorithms presented in [25]. The main difference is that we use h-sets which can have different stable dimensions. Unfortunately, the algorithms presented in [25] do not work in this case.

Let N, M be h-sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, such that $u(N) = u(M) = u$ and let $f : |N| \to \mathbb{R}^m$ be continuous. In order to prove that the covering relation $N \overset{f}{\Longrightarrow} M$ holds, it is necessary to find a homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$ and a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$ satisfying conditions (4–8).

A good candidate for the map A is an approximation of the derivative of $f_c = c_M \circ f \circ c_N^{-1}$ computed at zero and projected onto the unstable directions. Thus

$$A : \mathbb{R}^u \ni p \to \pi_u(Df_c(0)(p,0)) \in \mathbb{R}^u,$$

where $\pi_u : \mathbb{R}^n \to \mathbb{R}^u$ is a projection onto the first u variables. Notice, that there is not need to compute $Df_c(0)$ exactly. In fact, in the algorithms described below we can use an arbitrary linear map $A : \mathbb{R}^u \to \mathbb{R}^u$ which satisfies (8).

Now define a homotopy between f_c and $(p,q) \to (A(p),0)$ by

$$h(t,p,q) = (1-t)f_c + t(A(p),0), \quad \text{for } (p,q) \in \overline{B_u}(0,1) \times \overline{B_s}(0,1). \tag{47}$$

Obviously the homotopy (47) satisfies conditions (4) and (7). We need to check if the homotopy (47) satisfies conditions (5–6).

Definition 5.1. *Let $U \subset \mathbb{R}^n$ be a bounded set. We say that $\mathcal{G} \subset 2^{\mathbb{R}^n}$ is a partition of U if*

(1) *$\mathcal{G}$ is a finite set*
(2) *$U \subset \bigcup_{G \in \mathcal{G}} G$*

Definition 5.2. *An interval $[a,b]$ is called representable if the ends a and b are representable floating point numbers* [6].

Definition 5.3. *For $U \subset \mathbb{R}$ by $(U)_I$ we denote the interval enclosure of the set, i.e., the set $(U)_I$ is the smallest representable interval containing U.*

For $U \subset \mathbb{R}^n$ by $(U)_I$ we denote $(\pi_1(U))_I \times \cdots \times (\pi_n(U))_I$, where π_i is a projection onto the i-th variable.

Notice, that $\pm\infty$ are representable numbers, hence $(U)_I$ always exists. First we discuss how we can rigorously verify condition (5).

Algorithm 5.4.

```
function ComputeUnstableWall(G1 : partition, G2 : partition) : bool
var
    LX, Z : representable sets;
begin
    foreach G1 ∈ G1
    begin
        LX := (A(G1) × {0})_I;// 0 ∈ R^s(M)
        foreach G2 ∈ G2
        begin
            Z := (f_c(G1 × G2) ∪ LX)_I;
            if not π_u(Z) ⊂ R^u \ cl(B_u)(0,1) return false;
        end;
    end;
    return true;
end.
```

Lemma 5.5. *Assume N, M are h-sets in $\mathbb{R}^n, \mathbb{R}^m$, respectively and let $f : |N| \to \mathbb{R}^m$ be continuous. Let $\mathcal{G}_1$ be a partition of ∂B_u and let $\mathcal{G}_2$ be a partition of $\overline{B_s}$. If Algorithm 5.4 is called with arguments $(\mathcal{G}_1, \mathcal{G}_2)$ and returns* **true** *then the homotopy defined in (47) satisfies condition (5).*

Proof: Let $(p, q) \in N_c^-$. Since $\mathcal{G}_1$ is a partition of ∂B_u and $\mathcal{G}_2$ is a partition of $\overline{B_s}$,

$$\mathcal{G}_1 \times \mathcal{G}_2 := \{G_1 \times G_2 \mid G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2\}$$

is a partition of N_c^-. Therefore $(p, q) \in G_1 \times G_2$ for some $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$. Since Algorithm 5.4 stops and returns **true**, the condition

$$\pi_u((f_c(G_1 \times G_2) \cup LX)_I) \subset \mathbb{R}^u \setminus \overline{B_u} \tag{48}$$

is satisfied. An interval enclosure in the above formula gives a convex set, therefore $h(t, p, q) \in (f_c(G_1 \times G_2) \cup LX)_I$ for $t \in [0, 1]$. From this and from (48) we obtain that $h(t, p, q) \notin M_c$. □

Now we discuss how we verify condition (6). If the sets N and M have the same dimension it has been proven [25] that we can reduce computations to the boundary of set N_c. In the general case of different dimensions we need to verify some conditions on the whole set N_c.

Algorithm 5.6.

```
function ComputeImageOfSet(G1 : partition, G2 : partition) : bool
var
    LX, Z : representable sets;
begin
    foreach G1 ∈ G1
    begin
        LX := (A(G1) × {0})_I;// 0 ∈ R^s(M)
        foreach G2 ∈ G2
        begin
            Z := (f_c(G1 × G2) ∪ LX)_I;
            if (π_u(Z) ∩ cl(B_u) ≠ ∅ and π_s(M)(Z) ⊄ B_s(M)) return false;
        end;
    end;
    return true;
end.
```

Lemma 5.7. *Assume N, M are h-sets, and $f : |N| \to \mathbb{R}^m$ is continuous. Let $\mathcal{G}_1$ be a partition of $\overline{B_u}$ and $\mathcal{G}_2$ be a partition of $\overline{B_{s(N)}}$. If Algorithm 5.6 is called with arguments $\mathcal{G}_1$, $\mathcal{G}_2$ and returns* **true** *then the homotopy defined by (47) satisfies condition (6).*

Proof: Let $x \in N_c$. Then $x \in G_1 \times G_2$ for some $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$. Let Z be defined as in Algorithm 5.6. Since interval enclosure gives a convex set we know that $h(t,x) \in Z$ for all $t \in [0,1]$.

Since Algorithm 5.6 stops and returns **true** then either $\pi_u(Z) \cap \overline{B_u} = \emptyset$ or $\pi_{s(M)}(Z) \subset B_{s(M)}$. If $\pi_u(Z) \cap \overline{B_u} = \emptyset$ then

$$Z \cap \left(\overline{B_u} \times \overline{B_{s(M)}}\right) = Z \cap M_c = \emptyset$$

which implies that $h(t,x) \notin M_c$ for $t \in [0,1]$.

If $\pi_{s(M)}(Z) \subset B_{s(M)}$ then

$$Z \cap (\overline{B_u} \times \partial B_{s(M)}) = Z \cap M_c^+ = \emptyset.$$

This shows that $h(t,x) \notin M_c^+$ for $t \in [0,1]$. □

Remark 5.8. *In Algorithm 5.6 one could use a simpler condition*

if $(\pi_{s(M)}(Z) \not\subset B_{s(M)})$ **return false**;

Clearly, with this change Lemma 5.7 still holds true. However, this condition is usually too restrictive – see for example $N_7 \overset{P_c}{\Longrightarrow} N_8$ in Fig. 9. The projection of $P_c(|N_7|)$ onto the stable direction of $|N_8|$ has nonempty intersection with the projection of N_8^+ but the covering relation holds true.

Algorithms 5.4 and 5.6 show how to verify a covering relation for a map which does not depend on a parameter. In order to check the existence of a covering relation for a range of parameter values as it is required in Lemmas 4.1, 4.5 and 4.7 we find a partition $\mathcal{G}_c$ of $[c_{\min}, c_{\max}]$ and we use these algorithms for each element in the partition $\mathcal{G}_c$.

5.2. **How to estimate the unstable manifold of $x_-(c)$?** Recall that the map $W : |C| \to \Theta$ is defined as the second intersection of one branch of the unstable manifold of $x_-(c)$ with the Poincaré section.

The proof of Lemma 4.2 requires verification of two conditions

(1) W is well defined and continuous
(2) there exists the homotopy required for the existence of the covering relation $C \overset{W}{\Longrightarrow} N_4$

Now we present the main idea of a computer assisted proof that W is well defined. Let us fix $c \in |C|$. We define a very small three-dimensional h-set centered at $x_-(c)$ in the following way. Let $u(c)$, $s_1(c)$ and $s_2(c)$ be the eigenvectors of the linearized flow in $x_-(c)$. We can find an explicit formula for these vectors, namely

$$\begin{aligned}
u(c) &= 0.012\left(\frac{3\sqrt{2}Q_1(c)}{cQ_2(c)} + \frac{Q_2(c)}{6\sqrt{2}cQ_1(c)}, \frac{6Q_1(c)}{Q_2(c)}, 1\right)\\
s_1(c) &= 0.0024 \cdot \Re\left(\frac{-1}{\sqrt{2}cQ_3(c)} - \frac{Q_3(c)}{\sqrt{2}c}, \frac{-1}{Q_3(c)}, 1\right)\\
s_2(c) &= 0.0024 \cdot \Im\left(\frac{-1}{\sqrt{2}cQ_3(c)} - \frac{Q_3(c)}{\sqrt{2}c}, \frac{-1}{Q_3(c)}, 1\right)
\end{aligned} \tag{49}$$

where $\Re(z)$ is the real part and $\Im(z)$ is the imaginary part of the complex vector $z \in \mathbb{C}^3$ and

$$\begin{aligned} Q_1(c) &= \left(27\sqrt{2}c + \sqrt{108 + 1458c^2}\right)^{\frac{1}{3}} \\ Q_2(c) &= -6 \cdot 2^{\frac{1}{3}} + 2^{\frac{2}{3}} Q_1(c)^2 \\ Q_3(c) &= \left(1 + i\sqrt{3}\right)\left(2^{\frac{2}{3}} Q_1(c)\right)^{-1} + \left(1 - i\sqrt{3}\right) Q_1(c) \left(6 \cdot 2^{\frac{2}{3}}\right)^{-1} \end{aligned} \tag{50}$$

We define a three-dimensional set $H(c)$ built on these vectors but we change the role of the stable and unstable vectors, i.e. vector $u(c)$ is used as the stable direction and $s_1(c)$, $s_2(c)$ are used as the two unstable directions. More precisely, put $u(H(c)) = 2$, $s(H(c)) = 1$ and define

$$\begin{aligned} |H(c)| &= \{x_-(c) + p \cdot s_1(c) + q \cdot s_2(c) + r \cdot u(c) \mid p, q, r \in [-1, 1]\} \\ c_{H(c)} &: \mathbb{R}^3 \ni u \to M^{-1}(u - x_-(c)) \in \mathbb{R}^2 \times \mathbb{R} \end{aligned}$$

where $M = [s_1(c)^T, s_2(c)^T, u(c)^T]$ is a square matrix. It is important to note that $u(c)$, $s_1(c)$ and $s_2(c)$ are smooth functions of c for $c \in |C|$. Next, applying Algorithms 5.4 and 5.6 we prove that $H(c) \overset{\phi(c,-T_H,\cdot)}{\Longrightarrow} H(c)$ for $c \in |C|$, where $T_H = 1.4$. The reason for changing the role of stable and unstable eigenvectors is that we compute a backward trajectory on $|H(c)|$. Moreover, we verify that

$$\phi(c, [0, -T_H], |H(c)|]) \subset \{(x, y, z) \in \mathbb{R}^3 \mid x < 0\} \quad \text{for } c \in |C|.$$

Notice, that $H(c)^+$ consists of two connected walls and both of them form a horizontal disc in $H(c)$. Let us denote these discs by $b(c)^+$ and $b(c)^-$, where the sign corresponds to the signum of $u(c)$ on $H(c)^+$ in the $(s_1(c), s_2(c), u(c))$ coordinate system. More precisely,

$$b(c)^\pm(x, y) = c_{H(c)}^{-1}(x, y, \pm 1),$$

where $c_{H(c)}$ is the homemorphism in the definition of $H(c)$. From Corollary 3.10 applied to the sequence of covering relations

$$H(c) \overset{\phi(c,-T_H,\cdot)}{\Longrightarrow} H(c) \overset{\phi(c,-T_H,\cdot)}{\Longrightarrow} H(c) \cdots$$

and both horizontal $b(c)^\pm$ disc we obtain that there exist two points $w(c)^\pm \in b(c)^\pm(\overline{B_2})$ such that

$$\phi(c, -nT_H, w(c)^\pm) \in |H(c)| \quad \text{for} \quad n = 1, 2, \ldots$$

Next we use the energy function (36) and similar arguments as in Lemma 4.8 in order to prove that the α-limit set of $w(c)^\pm$ consists of a single point, which is an equilibrium point, i.e.

$$\lim_{t \to -\infty} \phi(c, t, w(c)^\pm) = x_-(c)$$

Thus for all parameter values $c \in |C|$ both connected components of $H(c)^+$ contain a point in the unstable manifold of $x_-(c)$.

Next we verify with the computer assistance that for all $c \in |C|$ the second intersection of $b(c)^+(\overline{B_2})$ with the Poincaré section Θ exists and it is transverse. This shows that $W(c)$ is well defined for $c \in |C|$.

In order to show that W is continuous let $P^2 : |C| \times \mathbb{R}^3 \dashrightarrow \Theta$ denote the second intersection of the solution $\phi(c, t, u)$ with Poincaré section Θ.

Lemma 5.9. *Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and let zero be a regular value of α. Let $\Theta = \alpha^{-1}(0)$ be a Poincaré section for the local flow ϕ induced by $f : \mathbb{R}^n \to \mathbb{R}^n$ and let $P : \mathbb{R}^n \to \Theta$*

be defined by

$$P(u) = \phi(t_P(u), u)$$
$$t_P(u) = \inf\{t > 0 \mid \alpha(\phi(t_P(u), u)) = 0\}$$

If the flow is transverse to Θ at $P(u)$, then P is smooth in some neighborhood of u.

Proof: Define $g = \alpha \circ \phi$ and let $(t, u) \in \mathrm{dom}(g)$. Since ϕ is a local flow we get that g is defined in some neighborhood of (t, u) and

$$\frac{\partial g}{\partial t}(t, u) = \nabla\alpha(\phi(t, u)) \cdot f(\phi(t, u)) \tag{51}$$

where $x \cdot y$ denotes a scalar product. Take $u \in \mathrm{dom}(P)$. Since $\nabla\alpha(P(u)) \cdot f(P(u)) \neq 0$, from (51) we have

$$\frac{\partial g}{\partial t}(t_P(u), u) = \nabla\alpha(\phi(t_P(u), u)) \cdot f(\phi(t_P(u), u)) \neq 0.$$

From implicit function theorem t_P is smooth, which implies that P is smooth in some neighborhood of (t, u). □

Lemma 5.9 applied to the flow

$$\begin{cases} \dot{c} = 0 \\ \dot{x} = y \\ \dot{y} = z \\ \dot{z} = c^2 - y - \frac{1}{2}x^2 \end{cases}$$

and the section given by $\alpha(c, x, y, z) = z$ implies that P^2 is smooth. Hence we obtain that W is smooth as the composition of a local parameterization of the center-unstable manifold of $x_-(c)$ with P^2.

It remains to prove that there exists a homotopy required for the existence of covering relation $C \overset{W}{\Longrightarrow} N_4$. Let us define

$$\widetilde{P} : |C| \times \overline{B_2} \ni (c, x) \to P^2(c, b(c)^+(x)) \to \Theta$$

The map $\widetilde{P}$ is continuous, because the vectors $u(c)$, $s_1(c)$ and $s_2(c)$ used to define $b(c)^+$ are smooth functions of c. Since the horizontal disc $b(c)^+$ is injective by its definition we have

$$W(c) = P^2(c, w(c)^+) = \widetilde{P}\left(c, (b(c)^+)^{-1}(w(c)^+)\right).$$

Applying Algorithms 5.4 and 5.6 one can prove that $C \overset{\widetilde{P}(\cdot,x)}{\Longrightarrow} N_4$ for $x \in \overline{B_2}$. We will show that this implies that $C \overset{W}{\Longrightarrow} N_4$. To avoid problems with notation put $Q = C_c = \overline{B_1}$ and let q be the homeomorphism from the definition of the h-set C.

Lemma 5.10. *If $C \overset{\widetilde{P}(\cdot,x)}{\Longrightarrow} N_4$ for all $x \in \overline{B_2}$ then $C \overset{W}{\Longrightarrow} N_4$.*

Proof: Let $h_1 : [0, 1] \times \overline{B_2} \times Q \to \mathbb{R}^2$ and $A : \mathbb{R} \to \mathbb{R}$ be the homotopy and the linear map resulting from Lemma 3.5 applied to $Z = \overline{B_2}$, $N = C$, $M = N_4$ and $f = \widetilde{P}$. We define the homotopy

$$h : [0, 1] \times Q \ni (t, c) \to h_1\left(t, (b(q^{-1}(c))^+)^{-1}(w(q^{-1}(c))^+), c\right) \in \mathbb{R}^2 \tag{52}$$

From (16) we have

$$\begin{aligned} h(0,c) &= c_{N_4}\left(\widetilde{P}\left(q^{-1}(c),(b(q^{-1}(c))^+)^{-1}(w(q^{-1}(c))^+\right)\right) \\ &= c_{N_4}\left(P^2\left(q^{-1}(c),w(q^{-1}(c))^+\right)\right) = c_{N_4}(W(q^{-1}(c))) \\ &= (c_{N_4}\circ W\circ q^{-1})(c) \end{aligned}$$

hence homotopy (52) satisfies (4). Conditions (17–18) imply that homotopy (52) satisfies (5–6). Finally, from (19) we know that

$$h(1,c) = A(c)$$

which shows that homotopy (52) satisfies (7–8). This completes the proof. □

5.3. **Technical data.** In order to compute Poincaré map P_c, $\widetilde{P}$ and time translations Φ_M, Φ_H, $\phi(c,-T_H,\cdot)$ we used the interval arithmetic [20], set algebra and the Lohner algorithm [15, 28] developed at the Jagiellonian University by CAPD group [1]. The C++ source files of the program with an instruction how it should be compiled and run are available at [22]. All computations were performed with the Pentium IV, 3GHz processor and 512MB RAM under Mandriva Linux LE 2005 with gcc-4.0.1 and MS Windows XP Professional with gcc-3.4.4. The computations took approximately 4 minutes.

6. Final remarks.

In this paper we show that the Michelson system (1) possesses very complicated homoclinic and heteroclinic structure in the range of parameter values $c \in [c_{\min}, c_{\max}]$. As it is observed in [16, 14] it seems that in some subinterval of $[c_{\min}, c_{\max}]$ the structure of heteroclinic and homoclinic solutions is richer. One observes two extra nonsymmetric periodic solutions which allow us to include two new symbols in the chaotic dynamics. These periodic orbits are presented in Fig.12. It seems that for these parametr values the homoclinic and heteroclinic dynamics may be richer.

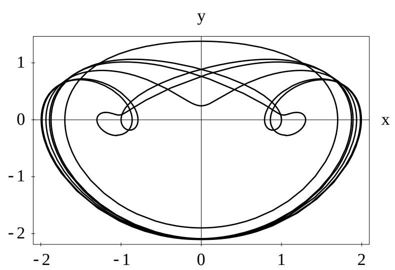

FIGURE 12. A six periodic orbits possibly existing in some subset of parameter values of $[c_{\min}, c_{\max}]$. Compare Fig.1.

The numerical simulation shows that for parameter values $c \approx 0.49$ there exists a second family of homoclinic and one-dimensional heteroclinic solutions. We believe the method introduced in this paper may provide a tool for verifying this conjecture. Examples of such heteroclinic and homoclinic orbits in this range of parameters are presented in Figs.13 and 14.

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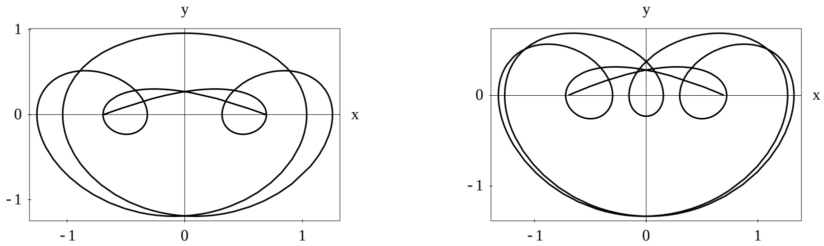

FIGURE 13. Heteroclinic solutions connecting $x_{-}(c)$ with $x_{+}(c)$ for parameter values $c \approx 0.48456$ and $c \approx 0.49476332$, respectively.

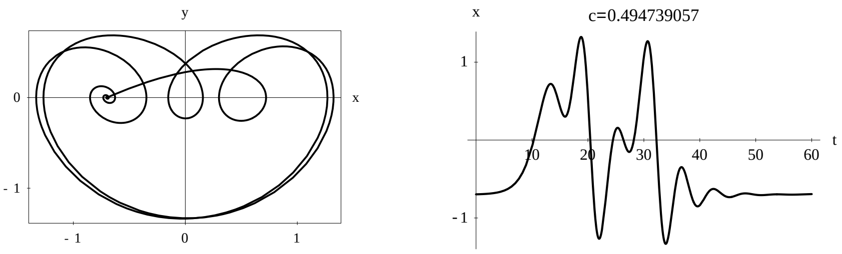

FIGURE 14. A Shilnikov homoclinic solution for $c \approx 0.494739057$.

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