Isolating neighborhoods and chaos

by

Konstantin Mischaikow[1,2] and Marian Mrozek [3]

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Abstract. We show that the map part of the discrete Conley index carries information which can be used to detect the existence of connections in the repeller-attractor decomposition of an isolated invariant set of a homeomorphism. We use this information to provide a characterization of invariant sets which admit a semi-conjugacy onto the space of sequences on K symbols with dynamics given by a subshift. These ideas are applied to the Henon map to prove the existence of chaotic dynamics on an open set of parameter values.

[1,] Center for Dynamical Systems and Nonlinear Studies, Georgia Institue of Technology, Atlanta GA 30332.

[2] Research funded in part by NSF Grant DMS-9101412

[3] Center for Dynamical Systems and Nonlinear Studies, Georgia Institue of Technology, Atlanta GA 30332; on leave from Computer Science Department, Jagiellonian University, Kraków, Poland.

1. Introduction

Examples of complicated or chaotic dynamics are ubiquitous; extending well beyond the mathematical literature into the realm of theoretical sciences and engineering. Unfortunately, the set of examples for which chaos has been rigorously demonstrated is quite small. This is not surprising since there are at least two major obstacles hindering verification. The first, and foremost is that this is a problem of global analysis. Thus, in most cases where chaotic dynamics has been proven, the strategy has involved analyzing a simple singular or integrable problem and then perturbing the results. An added difficulty arises from that fact that in order to insure an understanding of the dynamics after perturbation some form of transversality is needed. (see [Mo, GH]). The second difficulty is our inability to detect and control global bifurcations. Thus, there is no simple mechanism for determining when the chaotic dynamics of one system can be homotoped to another system.

These problems should be contrasted with an area of global analysis for which considerable progress has been made, namely, the existence theory for fixed points. The general strategy in this case is well known: find a simple system for which a fixed point can be found; determine that the degree of the fixed point is non-zero; and homotope to the system of interest. The method works because one can replace the geometric or analytic result of the existence of a fixed point with the algebraic quantity represented by the degree. The important point is that the degree is insensitive to bifurcations, i.e. as long as fixed points do not encroach upon the boundary of the domain of interest the degree remains constant. This is true even if fixed points appear or disappear along the homotopy. Thus a non–zero degree provides a lower bound (usually one) on the number of fixed points.

The purpose of this paper is to present an algebraic invariant which remains constant under appropriate homotopies and simultaneously provides a minimal description of the complexity of the dynamics of the system. Though, obviously, a complete understanding is the ideal result, for most applications a nontrivial lower bound on the structure of the dynamics is sufficient. With this in mind we have chosen to describe the dynamics in terms of semi-conjugacies. Estimates of this form by (implicitly given) gradient-like flows is the essence of Conley's theorem on the gradient structure of a flow (see [Co]). C. McCord and the first author proposed in [MMC] (see also [Mi], [Mi-Mo]) a technique based on the Conley index theory, which can provide semi-conjugacies onto explicitly given systems which are not necessarily gradient-like. In these results all the potential chaotic dynamics was intentionally collapsed onto periodic orbits. The point of this paper is to show that similar techniques can be used to obtain semi-conjugacies with subshift dynamics on a specified set of symbols. We restrict ourselves to the case of discrete dynamical systems since they appear to be simpler. The flow case is left for future investigation.

The algebraic invariant we will use is based on the Conley index for homeomorphisms as developed in [Mr2, Mr4] (see also [RS]). Recall that since the Conley index is an index for isolated invariant sets, it remains constant through out homotopies for which an isolating neighborhood can be preserved.

Smale's horseshoe provides the perfect example with which to explain our ideas. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a plane homeomorphism which transforms the square Q with vertices A, B, C, D into the horseshoe-shaped region $A'B'C'D'$ as indicated in Figure 1. The goal is to describe the dynamics on the maximal invariant set in Q, which is denoted by $\mathrm{Inv}(Q)$. Let the two components of $f^{-1}(Q) \cap Q$ be given by N_0 and N_1, and set $N := N_0 \cup N_1$. Observe that $\mathrm{Inv}(N) = \mathrm{Inv}(Q)$. Thus given $x \in \mathrm{Inv}(Q)$ its trajectory lies in $N = N_0 \cup N_1$, and hence, there is a well defined sequence of the symbols 0 and 1 which indicate how x travels through Q. This of course defines a map $\rho : \mathrm{Inv}(Q) \to \Sigma^2$, the space of bi–infinite sequences consisting of 0 and 1. For this example it is easy to check that ρ is a homeomorphism, and that if $\sigma : \Sigma^2 \to \Sigma^2$ denotes the shift map then

$$\rho \circ f = \sigma \circ \rho$$

i.e. ρ is a conjugacy.

Our aim is to use the Conley index to obtain algebraic invariants which guarantee the existence of a surjective map ρ. The first step in this direction is to identify the appropriate isolating neighborhoods. Consider the four components N_{00}, N_{01}, N_{10} and N_{11} of $N \cap f(N)$ as indicated in Figure 1. It is easy to check that any combination of these sets forms an isolating neighborhood. Furthermore:

- $\mathrm{Inv}(N_{00})$ and $\mathrm{Inv}(N_{11})$ consist of fixed points whose symbolic descriptions are given by the constant bi–infinite sequences of 0 and 1, respectively.

- $\mathrm{Inv}(N_{01}) = \emptyset = \mathrm{Inv}(N_{10})$. Any point whose trajectory leaves N_{00} and enters N_{11} passes through N_{01}. Similarly, trajectories from N_{11} to N_{00} pass through N_{10}.
- $(\mathrm{Inv}(N_{00}), \mathrm{Inv}(N_{11}))$ forms an attractor–repeller pair (in the sense of Conley [Co]) for $\mathrm{Inv}(N_{11} \cup N_{10} \cup N_{00})$; as does $(\mathrm{Inv}(N_{11}), \mathrm{Inv}(N_{00}))$ for $\mathrm{Inv}(N_{00} \cup N_{01} \cup N_{11})$.

Figure 1. U-horseshoe

This last point brings us to the issue of connecting orbits for attractor-repeller decompositions of an isolated invariant set. For both flows [Co] and maps [Mr3], it is known that the non–triviality of the boundary homomorphism for the exact sequence of the attractor–repeller pair implies the existence of connecting orbits. However, being a boundary map it can only detect the existence of connections when the difference in the index is of degree 1. In the case of the horseshoe the index of $\mathrm{Inv}(N_{00})$ is isomorphic to that of $\mathrm{Inv}(N_{11})$

and hence the boundary isomorphism must be zero. For maps, however, the index consists of both a graded module and an automorphism on this module. We show (Theorem 2.2) that the index automorphism for $\mathrm{Inv}(N_{00} \cup N_{01} \cup N_{11})$ can carry information concerning the existence of connecting orbits from $\mathrm{Inv}(N_{00})$ to $\mathrm{Inv}(N_{11})$.

As will be explained later these automorphisms can be viewed as matrices where the basis of the modules can be expressed in terms of the indexes of $\mathrm{Inv}(N_{00})$ and $\mathrm{Inv}(N_{11})$. In the simplest setting, i.e.

$$\mathrm{Con}^*(\mathrm{Inv}(N_{00})) \approx \mathrm{Con}^*(\mathrm{Inv}(N_{11})) \approx \mathbf{Z}_2,$$

there exist homomorphisms μ_{kl} which map the index of $\mathrm{Inv}(N_{ll})$ to the index of $\mathrm{Inv}(N_{kk})$. Thus a composition of these isomorphism, e.g.

$$\mu_{11} \circ \mu_{10} \circ \mu_{01} \circ \mu_{11} \circ \mu_{10}$$

is well defined and the proof (presented in Section 6) of the main theorem (Theorem 2.3) is now reduced to showing that if each μ_{kl} is an isomorphism then this implies the existence of a set of orbits which pass through the sets N_0 and N_1 in the order

$$N_1 \to N_1 \to N_0 \to N_1 \to N_1 \to N_0.$$

To do this we need to make observations concerning the Alexander–Spanier cohomology theory (Section 3).

However, at the level of generality considered in this paper there is no naturally defined homomorphism μ_{kl}. While the explanation of this lack of naturality is very simple, its importance cannot be overstated. The algebra of the Conley index is obtained from the topology of the dynamics via the *direct limit functor* L (see Section 4). This involves a loss of information. On the other hand, it is the index which continues rather than the topology. Therefore, from the point of view of applicability, the assumptions and hypothesis must be stated in terms of the algebra, while the conclusions should be stated in terms of the dynamics. Thus one of the essential problems is how does one lift the algebraic information back to the topological or geometrical information.

Returning to the problem at hand, concerning connecting orbits from $\mathrm{Inv}(N_{ll})$ to $\mathrm{Inv}(N_{kk})$, we conjecture that the relevant information from $\mathrm{Con}^*(\mathrm{Inv}(N_{ll}))$ to $\mathrm{Con}^*(\mathrm{Inv}(N_{kk}))$ is carried by the linear conjugacy classes of the homomorphisms between the spaces. In this paper we circumvent this problem by introducing the *connection number* α_{lk} which replaces the homomorphism μ_{kl} in the argument above.

Before turning to the next section, where precise statements of our results can be found, we wish to emphasize that the geometric ideas which underlie our algebraic techniques are by now well understood. A careful description of how geometric and analytic methods can be used to construct a conjugacy ρ can be found in Moser[1] and leads to two observations:

1. the general idea for constructing the map onto the sequence of symbols involves showing that the forward and backward images of carefully chosen subsets of phase space intersect in a transverse fashion;
2. the homeomorphism is obtained by imposing conditions on the rates of contraction and expansion of the sets.

Implicit in the existence of our isomorphisms between the graded modules of the Conley indexes is some form of transversality. The advantage to the algebraic approach is that this transversality whether topological or differential need not be checked explicitly. On the other hand, eliminating the use of analysis precludes an estimate on the expansion and contraction rates, and hence, there is no hope of strengthening the semi-conjugacy to a conjugacy, i.e. we have explicitly given up on an exact description of the dynamics.

One final comment is in order. The Conley index of an isolated invariant set is computed by choosing an isolating neighborhood and quotienting the exit set to a point. Obviously, information is lost in this process. In discussions Conley emphasized the importance of understanding how to capture this information. (See [Co II.2.4] for an illuminating example and [E2] for a concrete description in the setting of maps). In the setting of discrete dynamical systems our results can be viewed as a partial answer to this problem, though in the case of flows the problem remains essentially untouched.

[1] For a description of the geometric ideas explicitly stated in terms of isolating neighborhoods and how this leads to the mapping ρ see [E1].

To explain what we mean by this, let us return to the example of the Smale horseshoe. As presented above, Q is the natural isolating neighborhood for the horseshoe. However, the Conley index of $\mathrm{Inv}(Q)$ is trivial from which one can draw no conclusions concerning even the existence of an invariant set for Q. The results of this paper imply the following: subisolating neighborhoods and their indexes provide information about the dynamics on the global isolated invariant set. Obviously, given an infinite sequence of isolating neighborhoods which converge down onto the invariant set and knowledge of the various indexes, one would expect to be able to provide a highly detailed description of the dynamics. However, obtaining this information directly appears to be unreasonably difficult for nontrivial systems (one can interpret Theorem 2.4 as a means of obtaining such information, but by continuing the flow to an easily understood example where the complete information is known). Furthermore, and this is the point of this paper (Theorem 2.3), non-trivial lower bounds on the complexity of the dynamics can be obtained from a finite amount of information. An open question is how much redundancy exists in the index information that can be computed, i.e. do there exist relationships between the various indexes so that non-triviality of some indexes forces others to be non-trivial (like in the case of Morse decompositions for which one has the existence of connection matrices).

The organization of the paper is as follows. Our main results are presented in Section 2. In Section 3 we briefly recall some basic facts concerning the Alexander-Spanier cohomology. Isolating neighborhoods and the Conley index for maps are summarized in Section 4. Proofs of preliminary results and of the main theorems are presented in Sections 5 and 6. (The reader not familiar with the Conley index theory will probably prefer to study the sections in the order 1,3,4,2,5,6,7.) Finally, we conclude in Section 7 with an example which is meant to indicate:

1. how the Conley indexes can be computed for a specific dynamical system, and
2. the need for additional theorems of the type presented here.

The dynamical system we have chosen to use is the Henon map. Though we believe that the results presented here are new, it must be emphasized that it is the techniques (both the strengths and weaknesses, thereof) rather than the results which are the focus of this section. As we hope will be made clear, it is premature to believe that a systematic study of the Henon attractor should be undertaken with the current set of index tools.

2. Main results.

Let $f : X \to X$ be a homeomorphism of a locally compact metric space X. $N \subset X$ will always denote an isolating neighborhood under f which decomposes into K disjoint compact subsets $\{N_k \mid k = 1, \ldots, K\}$. Let

$$\mathcal{A} = \left[\alpha_{ij}\right]$$

be a $K \times K$ matrix with entries $\alpha_{ij} \in \{0, 1\}$. Define

$$\Sigma_{\mathcal{A}} \subset \prod_{-\infty}^{\infty} \{1, \ldots, K\}$$

by

$$\Sigma_{\mathcal{A}} = \{a = (\ldots, a_{-1}, a_0, a_1, \ldots) \mid \alpha_{a_n a_{n+1}} = 1\}$$

and let $\sigma : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}$ be given by

$$(\sigma(a))_n = a_{n+1}.$$

Let $\Sigma_{\mathcal{A}}^n \subset \Sigma_{\mathcal{A}}$ denote the collection of all subsequences of length n in $\Sigma_{\mathcal{A}}$ and let Σ_K^n denote all subsequences of length n on K symbols. Given $a = (a_0, \ldots, a_n) \in \Sigma_{\mathcal{A}}^n$ set

$$N_a := \bigcap_{i=0}^{n} f^i(N_{a_i})$$

and given $a = (a_0, \ldots, a_n) \in \Sigma_K^n \setminus \Sigma_{\mathcal{A}}^n$ set $N_a = \emptyset$. For any $A \subset \Sigma_K^n$ define

$$N_A := \bigcup_{a \in A} N_a.$$

To simplify the notation we shall set

$$S_A := \mathrm{Inv}(N_A).$$

The following result is trivial. However, because of its importance we elevate it to the status of a theorem.

Theorem 2.1. *For any $A \subset \Sigma_K^n$ the set N_A is an isolating neighborhood for f.*

For each isolating neighborhood N_A its Conley index

$$\mathrm{Con}^*(N_A) = (CH^*(N_A), \chi^*(N_A))$$

is defined. Recall [Mr], that it is a pair consisting of a graded module $CH^*(N_A)$ over a fixed commutative ring with unity $\mathbf{G}$ and an isomorphism $\chi^*(N_A) : CH^*(N_A) \to CH^*(N_A)$.

It follows from Theorem 2.1 that N_k, N_l and $N_{\{kk,kl,ll\}}$ are isolating neighborhoods. This sets up the following theorem which relates the index automorphism to the existence of connecting orbits.

Theorem 2.2.

(i) (S_k, S_l) *is an attractor-repeller pair in* $S_{\{kk,kl,ll\}}$.

(ii) *The Conley indices of S_k, S_l and $S_{\{kk,kl,ll\}}$ are related by the following commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & CH^*(S_l) & \longrightarrow & CH^*(S_{\{kk,kl,ll\}}) & \longrightarrow & CH^*(S_k) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \chi(S_l)} & & \Big\downarrow{\scriptstyle \chi(S_{\{kk,kl,ll\}})} & & \Big\downarrow{\scriptstyle \chi(S_k)} & & \\ 0 & \longrightarrow & CH^*(S_l) & \longrightarrow & CH^*(S_{\{kk,kl,ll\}}) & \longrightarrow & CH^*(S_k) & \longrightarrow & 0 \end{array} \tag{1}$$

in which vertical arrows denote the corresponding index maps.

(iii)

$$CH^*(S_{\{kk,kl,ll\}}) = CH^*(S_k) \oplus CH^*(S_l). \tag{2}$$

(iv) If

$$\chi(S_{\{kk,kl,ll\}}) \text{ and } \chi(S_k) \oplus \chi(S_l) \text{ are not conjugate} \tag{3}$$

then there exists a connecting orbit from S_l to S_k in $S_{\{kk,kl,ll\}}$, i.e. an element $x \in S_{\{kk,kl,ll\}}$ such that $\alpha(x) \subset S_k$ and $\omega(x) \subset S_l$.

Condition (3) is equivalent to assuming that every maximal non-decomposable invariant subspace of $\chi(S_k)$ when embedded into $CH^*(S_{\{kk,kl,ll\}})$ is also maximal with respect to $\chi(S_{\{kk,kl,ll\}})$. This can be verified by comparing the Jordan forms of $\chi(S_k)$ and $\chi(S_{\{kk,kl,ll\}})$.

Since both isomorphisms in (3) are graded isomorphisms, they fail to conjugate if they do so in one of the components. Thus consider the special case when for some fixed $m \in \mathbf{N}$

$$\mathrm{Con}^m(S_l) \approx \mathrm{Con}^m(S_1) \approx (\mathbf{G}, \mathrm{id}). \tag{4}$$

It follows then from (1) that in matrix form

$$\chi^k(S_{\{kk,kl,ll\}}) = \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}$$

in which μ can depend on the choice of the complement to $CH^m(S_k)$ in the decomposition (2) given by (1). However, since all such matrices which represent the same map are conjugate, the number

$$\alpha_{kl} := \begin{cases} 0 & \text{if } \mu = 0 \\ 1 & \text{if } \mu \neq 0 \end{cases}$$

is an invariant of the attractor-repeller pair (S_k, S_l) in $S_{\{kk,kl,ll\}}$.

We shall call α_{kl} a *connection number*, because $\alpha_{kl} \neq 0$ implies $\chi^m(S_{\{kk,kl,ll\}})$ and $\chi^m(S_k) \oplus \chi^m(S_l)$ are not conjugate. Hence, by Theorem 2.2, if $\alpha_{kl} = 1$, then there exists a connecting orbit from S_l to S_k (compare with Example 5.5 in [RSZ]).

The G-horseshoe as presented in Figure 2 satisfies condition (4) with $m = 1$ and it is easy to verify that both connection numbers α_{12} and α_{21} are one. The U-horseshoe in Figure 1 fails to satisfy (4) when the coefficients are rationals, because then $\chi^1(S_1) = -\mathrm{id}$. However it does satisfy (4) when we take $\mathbf{Z}_2$ coefficients. Hence also in this case we have well defined connection numbers $\alpha_{21} = \alpha_{12} = 1$.

Theorem 2.2 illustrates that some information is lost when the discrete Conley index is considered as a zeta function (see [RSZ]) or as a collection of the eigenvalues of $\chi(S)$.

The following theorem provides the semi-conjugacy and the subshift dynamics used to describe $\mathrm{Inv}(N)$.

Theorem 2.3. *Let N be an isolating neighborhood. Assume that N is the disjoint union of compact sets $\{N_k \mid k = 1, \ldots, K\}$ and that for $k = 1, \ldots, K$*

$$\mathrm{Con}^m(S_k) \approx \mathrm{Con}^m(\mathrm{Inv}(N_k)) \approx (\mathbf{G}, \mathrm{id}).$$

Let α_{kl} denote the connection number obtained from $N_{\{kk,kl,ll\}}$ and set $\alpha_{kk} = 1$. Define $\mathcal{A} = \left[\alpha_{kl}\right]$ to be the resulting $K \times K$ matrix. If $\alpha_{kl} = 0$ implies that there are no connecting orbits from S_l to S_k, then there exists $d \in \mathbf{N}$ and a continuous surjection

$$\rho : \mathrm{Inv}\ (N) \to \Sigma_{\mathcal{A}}$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Inv}(N) \arrow[r, "f^d"] \arrow[d, "\rho"] & \mathrm{Inv}(N) \arrow[d, "\rho"] \\
\Sigma_{\mathcal{A}} \arrow[r, "\sigma"] & \Sigma_{\mathcal{A}}
\end{tikzcd}$$

Figure 2. G-horseshoe

The horseshoes in Figure 1 and in Figure 2 were models for Theorem 2.3. To apply this theorem to those examples, let $K = 2$ and observe that

$$\mathcal{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

This gives the desired semi-conjugacy onto the full shift dynamics on two symbols.

Remark: The assumption $\alpha_{kl} = 0$ implies that there are no connecting orbits from S_l to S_k in Theorem 2.3 is necessary, and shows that the algebraic information is not always sufficient. This may pose a serious obstacle in some applications. On the other hand, as was shown for the horseshoe, in the case of a full shift this assumption is vacuously true. Furthermore, the hypothesis that connecting orbits do not exist within an isolating neighborhood is fundamentally different from assuming the existence of connecting orbits. To show that connecting orbits do not exist it is sufficient to show that a section to the unstable set of the repeller is mapped out of the isolating neighborhood after a *finite* number of iterations. Thus, it is conceivable that for specific problems this condition could be verified.

The fact that d must be chosen from the set of natural numbers, rather than the "optimal" $d = 1$, arises from algebraic considerations in the proof. What this means geometrically is still not clear. In particular, the minimal possible d appears to depend upon an interplay between the subshift with which one wishes to define the semi–conjugacy and the labeling of the components of the isolating neighborhood. Under stronger assumptions one can show that $d = 1$ and that the pre-image of periodic orbits in the shift dynamics contains true periodic orbits of the original dynamical system. As an example of this consider the following theorem.

Theorem 2.4. *Assume $M \subset \mathbf{R}^n = \mathbf{R}^2 \oplus \mathbf{R}^{n-2}$ is a disjoint sum of two its compact subsets M_0, M_1, $N_i := M_i \cap \mathbf{R}^2$ and $f_t : \mathbf{R}^n \to \mathbf{R}^n$, ($t \in [0,1]$) is a homotopy. If N is an isolating neighborhood for all f_t and f_0 is a U- (or G-) horseshoe in $\mathbf{R}^n$, i.e. it decomposes into a homeomorphism with a globally attracting fixed point in $\mathbf{R}^{n-2}$ and a U- (or G-) horseshoe as in Fig.1 or in Fig.2 in $\mathbf{R}^2$ then there exists a continuous surjection*

$$p : \operatorname{Inv} N \to \Sigma_2$$

such that the following diagram

$$\begin{array}{ccc} \operatorname{Inv} N & \xrightarrow{f_1} & \operatorname{Inv} N \\ \downarrow p & & \downarrow p \\ \Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \end{array}$$

commutes. Moreover, for every periodic sequence $\alpha \in \Sigma_2$ the set $p^{-1}(\alpha)$ contains a periodic point of f_1 with the same period as α.

Assumptions of Theorem 2.4 are essentially stronger than those in Theorem 2.3. To see this consider a homeomorphism $\mathbf{R}^2 \times S^1 \to \mathbf{R}^2 \times S^1$ defined as a product of the G-horseshoe homeomorphism in Fig. 2 and the identity on S^1. Take $M_i := N_i \times S^1$, where $N_i, (i = 0, 1)$ is defined as in Figure 2. Then a direct computation shows that

$$\mathrm{Con}^n(M_0) = \mathrm{Con}^n(M_1) = \begin{cases} 0 & \text{for } n \notin \{1,2\} \\ (\mathbf{G}, \mathrm{id}) & \text{for } n \in \{1,2\} \end{cases} \tag{5}$$

and both connection numbers can be easily shown to be one. Hence the conclusion of Theorem 2.3 applies to f and also to any small perturbation of f. However (5) shows that there exists no homotopy to a G-horseshoe (or U-horseshoe) so that Theorem 2.4 is not applicable. The above example also shows that the assumptions of Theorem 2.3 cannot imply the existence of periodic points (replace the identity on S^1 with irrational rotation).

We finish this section with the following conjecture.

Conjecture 2.5. *Let $N = N_1 \cup N_2$ be an isolating neighborhood where N_1 and N_2 are disjoint compact sets. Assume that*

$$\mathrm{Con}^m(S_k) \approx \mathrm{Con}^m(\mathrm{Inv}(N_k)) \approx (\mathbf{G}, \mathrm{id}).$$

and that

$$\mathrm{Con}^*(\mathrm{Inv}(N)) \approx 0.$$

Then, there exists $d \in \mathbf{N}$ and a continuous surjection

$$\rho : \mathrm{Inv}\ (N) \to \Sigma_{\mathcal{A}}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Inv}(N) & \xrightarrow{f^d} & \mathrm{Inv}(N) \\ \downarrow \rho & & \downarrow \rho \\ \Sigma_{\mathcal{A}} & \xrightarrow{\sigma} & \Sigma_{\mathcal{A}} \end{array}$$

3. Alexander–Spanier Cohomology

In this section definitions and auxiliary results concerning the Alexander–Spanier cohomology are summarized. Let $\mathbf{G}$ be a fixed commutative ring with unity. For a topological space X and $n \in \mathbf{N}$, let $\Gamma^n(X)$ denote the $\mathbf{G}$-module of all functions $\varphi : X^{n+1} \to \mathbf{G}$. The coboundary homomorphism $\delta : \Gamma^n(X) \to \Gamma^{n+1}(X)$ is defined by the formula

$$\delta\varphi(x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \varphi(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}).$$

Since $\delta\delta = 0$, $\Gamma^*(X) := (\Gamma^n(X), \delta)$ is a cochain complex over $\mathbf{G}$. For $\varphi \in \Gamma^n(X)$ define the support of φ by

$$|\varphi| := \{x \in X \mid \forall\ V \subset X \text{ open } x \in V \Rightarrow \exists\ x_0, \ldots, x_n \in V \text{ s.t. } \varphi(x_0, \ldots, x_n) \neq 0\}.$$

Then, $\Gamma_0^n(X) := \{\varphi \in \Gamma^n(X) \mid |\varphi| = \emptyset\}$ is a submodule of $\Gamma^n(X)$ and $\Gamma_0^*(X) := (\Gamma_0^n(X), \delta)$ is a cochain subcomplex of $\Gamma^*(X)$. The quotient cochain complex $\Gamma^*(X)/\Gamma_0^*(X)$ is denoted by $\bar{\Gamma}^*(X)$. If $u \in \bar{\Gamma}^*(X)$ then $|\varphi|$ does not depend on φ for all representations $\varphi \in u$. Hence one can set $|u| := |\varphi|$, where $\varphi \in u$.

If Y is another topological space and $g : X \to Y$ is continuous then there is an induced cochain map $g^\# : \Gamma^*(Y) \to \Gamma^*(X)$ defined by the formula

$$(g^\#\varphi)(x_0, \ldots, x_n) = \varphi(g(x_0), \ldots, g(x_n)).$$

Since $g^\#(\Gamma_0^*(Y)) \subset \Gamma_0^*(X)$, there is also an induced map $g^\# : \bar{\Gamma}^*(Y) \to \bar{\Gamma}^*(X)$.

If $i : X_0 \hookrightarrow X$ is an inclusion, then $\bar{\Gamma}^*(X, X_0) := \ker i^\#$ is a cochain subcomplex of $\bar{\Gamma}^*(X)$. The Alexander–Spanier cohomology of the pair (X, X_0) is defined as the homology of the cochain subcomplex $\bar{\Gamma}^*(X, X_0)$ and denoted by $\bar{H}^*(X, X_0)$. If (Y, Y_0) is another topological pair and $\kappa : \bar{\Gamma}^*(Y, Y_0) \to \bar{\Gamma}^*(X, X_0)$ is a cochain map then it induces the map $\kappa^* : \bar{H}^*(Y, Y_0) \to \bar{H}^*(X, X_0)$. In particular if $g : (X, X_0) \to (Y, Y_0)$ is a continuous map then one has a map $(g^\#)^* : \bar{H}^*(Y, Y_0) \to \bar{H}^*(X, X_0)$, which for the sake of simplicity will be denoted by g^*.

The following four propositions are easy to verify.

Proposition 3.1. (see [Ma,p.229]) *Assume* $\varphi \in \Gamma^*(X)$. *Then*

$$f(|f^{\#}(\varphi)|) \subset |\varphi|.$$

Proposition 3.2. (see [Ma,p.227]) *Assume* $\varphi \in \Gamma^*(X)$. *Then*

$$[\varphi] \neq 0 \quad \Rightarrow \quad |\varphi| \neq \emptyset.$$

Assume $X_0, X_1 \subset X$ and $\varphi \in \Gamma^p(X, X_0 \cap X_1)$. For $i = 0, 1$ define

$$\kappa_{X_i}(\varphi)(x_0, \ldots, x_p) := \begin{cases} \varphi(x_0, \ldots, x_p) & \text{if } x_0, \ldots x_p \in X_{1-i} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 3.3. *If* X_0, X_1 *are closed subsets of* X *then* $\kappa_{X_i} : \Gamma^*(X, X_0 \cap X_1) \ni \varphi \to \kappa_{X_i}(\varphi) \in \Gamma^*(X, X_i)$ *is a well defined cochain map which also induces a cochain map*

$$\bar{\kappa}_{X_i} : \bar{\Gamma}^*(X, X_0 \cap X_1) \to \bar{\Gamma}^*(X, X_i)$$

Proposition 3.4. *Assume* V_0, V_1, V *are* **G**-*modules* $\iota_{V_i} : V_i \to V$, $p_{V_i} : V \to V_i$ *for* $i = 0, 1$ *are linear maps such that*

$$p_{V_i} \circ \iota_{V_j} = \begin{cases} id_{V_j} & i = j \\ 0 & i \neq j \end{cases} \tag{6}$$

and

$$\iota_{V_0} \circ p_{V_0} + \iota_{V_1} \circ p_{V_1} = id_V \tag{6'}$$

Then

$$(\iota_{V_0} \oplus \iota_{V_1}) : V_0 \oplus V_1 \to V \tag{7}$$

is an isomorphism with inverse (p_{V_0}, p_{V_1}).

Proposition 3.5. *Assume* $X_0, X_1 \subset X$ *are closed subsets of* X *such that*

$$X = \operatorname{int} X_0 \cup \operatorname{int} X_1.$$

If $\iota_{X_i} : (X, X_0 \cap X_1) \to (X, X_i)$ $(i = 0, 1)$ denote inclusions, then

$$\iota^{\#}_{X_i} : \bar{\Gamma}^*(X, X_i) \to \bar{\Gamma}^*(X, X_0 \cap X_1)$$

are also inclusions and

$$\bar{\Gamma}^*(X, X_0 \cap X_1) = \bar{\Gamma}^*(X, X_0) \oplus \bar{\Gamma}^*(X, X_1)$$

with the corresponding projections $\bar{\kappa}_{X_0}$ and $\bar{\kappa}_{X_1}$. Moreover

$$\bar{H}^*(X, X_0 \cap X_1) \cong \bar{H}^*(X, X_0) \oplus \bar{H}^*(X, X_1)$$

with the corresponding inclusions $\iota^*_{X_0}, \iota^*_{X_1}$ and projections $\kappa^*_{X_0}, \kappa^*_{X_1}$. Furthermore, if Y_0, Y_1 are closed subsets of Y such that $Y = \operatorname{int} Y_0 \cup \operatorname{int} Y_1$ and

$$\mu : \bar{\Gamma}^*(X, X_0 \cap X_1) \to \bar{\Gamma}^*(Y, Y_0 \cap Y_1)$$

is a cochain endomorphism then μ decomposes into a matrix of cochain maps

$$\mu = \begin{bmatrix} \mu_{00} & \mu_{01} \\ \mu_{10} & \mu_{11} \end{bmatrix}, \quad \text{where } \mu_{ij} = \bar{\kappa}_{Y_i} \circ \mu \circ \iota^{\#}_{X_j}.$$

Also,

$$\mu^* = \begin{bmatrix} \mu^*_{00} & \mu^*_{01} \\ \mu^*_{10} & \mu^*_{11} \end{bmatrix}, \quad \text{where } \mu^*_{\mathrm{ij}} = \kappa^*_{\mathrm{Y_i}} \circ \mu^* \circ \iota^*_{\mathrm{X_j}}.$$

Proof. The assumptions of Proposition 3.4 need to be verified. First, consider (6). Fix $i, j \in \{0, 1\}$, take $\varphi \in \Gamma^p(X, X_j)$ and set

$$\varphi_{ij} := \begin{cases} \varphi & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

It needs to be shown that $\kappa_{X_i}\varphi - \varphi_{ij}$ is locally zero. Choose $x \in X$. If $x \in \operatorname{int} X_j$ then there exists a neighborhood V of x such that $V \subset X_j$ and $\varphi|_V = 0$. Since $(\kappa_{X_i}\varphi - \varphi_{ij})(x_0, \ldots, x_p) \in \{\pm\varphi(x_0, \ldots, x_p), 0\}$, it follows that $(\kappa_{X_i}\varphi - \varphi_{ij})|_V = 0$. Thus, assume $x \in \operatorname{int} X_{1-j}$. Choose a neighborhood V of x such that $V \subset X_{1-j}$. One can modify V so that, in addition, $\varphi|_{V\cap X_j} = 0$ if $x \in X_j$ and $V \cap X_j = \emptyset$ if $x \notin X_j$. Then, $\kappa_{X_j}\varphi|_V = \varphi|_V$. Furthermore, if $i = j$ then $\varphi_{ii}|_V = \varphi|_V$, and hence, $\kappa_{X_j}\varphi|_V - \varphi|_V = 0$. Assume $i \neq j$ and choose $x_0, \ldots, x_p \in V$. If $x_0, \ldots, x_p \in X_j$ then $(\kappa_{X_i}\varphi - \varphi_{ij})(x_0, \ldots, x_p) = \varphi(x_0, \ldots, x_p) = 0$; otherwise $(\kappa_{X_i}\varphi)(x_0, \ldots, x_p) = 0$. Therefore, $(\kappa_{X_i}\varphi - \varphi_{ij})|_V = 0$.

Take in turn $\chi \in \Gamma^p(X, X_0 \cap X_1)$. Then,

$$\left[\left(\iota^{\#}_{X_0} \oplus \iota^{\#}_{X_1}\right) \circ (\kappa_{X_0}, \kappa_{X_1})\right](\chi) = \kappa_{X_0}(\chi) + \kappa_{X_1}(\chi).$$

In order to prove $(6')$ it needs to be shown that $\kappa_{X_0}(\chi) + \kappa_{X_1}(\chi) - \chi$ is locally zero. To this end take $x \in X$. Assume first that $x \in \operatorname{int} X_0$. Choose V a neighborhood of x such that $V \subset X_0$. Without loss of generality it can also be assumed that

$$V \cap X_1 = \emptyset \quad \text{if } x \notin X_1$$

or

$$\chi|_{V \cap X_0 \cap X_1} = 0 \qquad \text{otherwise.}$$

Choose $x_0, \ldots, x_p \in V$. Then

$$\kappa_{X_0}(\chi)(x_0, \ldots, x_p) = 0 \text{ and } \kappa_{X_1}(\chi)(x_0, \ldots, x_p) = \chi(x_0, \ldots, x_p),$$

hence

$$(\kappa_{X_0}(\chi) + \kappa_{X_1}(\chi) - \chi)|_V = 0.$$

A similar argument in case $x \in \operatorname{int} X_1$ finishes the proof of $(6')$. The remaining part of the proposition is obvious. $\square$

4. Isolated Invariant Sets and the Conley Index for Maps

Assume X is a locally compact metric space and $f : X \to X$ is a homeomorphism. A subset $S \subset X$ is called invariant (with respect to f) if $f(S) = S$. The maximal invariant subset of a given set $N \subset X$ will be denoted by $\mathrm{Inv}(N, f)$ or $\mathrm{Inv}\, N$, i.e.

$$\mathrm{Inv}\, N = \{x \in N |\ f^n(x) \in N \text{ for all } n \in \mathbf{Z}\}.$$

Similarly one defines $\mathrm{Inv}\, N^+ = \{x \in N |\ f^n(x) \in N \text{ for all } n \in \mathbf{Z}^+\}$ and $\mathrm{Inv}\, N^- = \{x \in N |\ f^n(x) \in N \text{ for all } n \in \mathbf{Z}^-\}$. A compact set $N \subset X$ is called an isolating neighborhood if $\mathrm{Inv}\ N \subset \mathrm{int}\, N$. The following statement is a triviality.

Proposition 4.1. *If N is an isolating neighborhood then also for any $n \in \mathbf{N}$,*

$$\bigcap_{i=0}^{n} f^i(N)$$

is an isolating neighborhood and $\mathrm{Inv}\, (\bigcap_{i=0}^{n} f^i(N)) = \mathrm{Inv}\ N$.

Assume N is an isolating neighborhood.

Definition 4.2. $P = (P_1, P_2)$ will be called an index pair for f in N if P_1, P_2 are closed subsets of N with $P_2 \subset P_1$ and

(i) $P_i \cap f^{-1}(N) \subset f^{-1}(P_i)$, $i = 1, 2$

(ii) $P_1 \backslash P_2 \subset f^{-1}(N)$

(iii) $\mathrm{Inv}\ N \subset \mathrm{int}(P_1 \backslash P_2)$

Additionally, if

(iv) $\mathrm{cl}(P_1 \backslash P_2) \cap \mathrm{cl}(f(P_2) \backslash P_1) = \emptyset$, we call the index pair P regular.

The following proposition follows easily from the proof of Lemma 4 in [Sp, p. 309]

Proposition 4.3. *If P is a regular index pair then $i_P : (P_1, P_2) \to (P_1 \cup f(P_2), P_2 \cup f(P_2))$ induces an isomorphism $i^{\#} : \bar{\Gamma}^*(P_1 \cup f(P_2), P_2 \cup f(P_2)) \to \bar{\Gamma}^*(P_1, P_2)$ and*

$$(i_P^{\#,p})^{-1}([\varphi]) = [\lambda_P(\varphi)]$$

where $\lambda_P : \Gamma^*(P_1, P_2) \to \Gamma^*(P_1 \cup f(P_2), P_2 \cup f(P_2))$ is defined by

$$\lambda_P(\varphi)(x_0, \ldots, x_p) = \begin{cases} \varphi(x_0, \ldots, x_p) & \textit{if } x_0, \ldots, x_p \in P_1 \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 4.4. The cochain map $f^{\#} \circ (i^{\#})^{-1}$ will be called the cochain index map and will be denoted by $f_P^{\#}$. The cohomology endomorphism $f_P^* := (f_P^{\#})^*$ will be called the (cohomology) index map.

The following definition of an isolating block for maps was proposed in [E2].

Definition 4.5. The compact set $N \subset X$ is called an isolating block for f if $f^{-1}(N) \cap N \cap f(N) \subset \operatorname{int} N$.

Proposition 4.6. *Assume N is an isolating neighborhood and $M \subset N$ is both open and closed in N. Then also M is an isolating neighborhood. Moreover, if N is an isolating block then so is M.*

Proof. Under the assumptions of our proposition we have $\operatorname{Inv} M \subset M \cap \operatorname{Inv} N \subset M \cap \operatorname{int} N = M \cap (\operatorname{int} M \cup \operatorname{int}(N \backslash M)) = \operatorname{int} M$. If N is an isolating block then $f^{-1}(M) \cap M \cap f(M) \subset M \cap \operatorname{int} N = \operatorname{int} M$. □

For an isolating block N we define $N^- := N \backslash f^{-1}(\operatorname{int} N)$ and call it the exit set.

Proposition 4.7. *Each isolating block is an isolating neighborhood. If N is an isolating block then (N, N^-) is a regular index pair for f in N.*

Proof. $\operatorname{Inv} N = \bigcap_{k=-\infty}^{\infty} f^k(N) \subset \bigcap_{k=-1}^{1} f^k(N) \subset \operatorname{int} N$, hence N is an isolating neighborhood. Obviously N^-, N are compact subsets of N and $N^- \subset N$. To show that $N^- \cap f^{-1}(N) \subset f^{-1}(N^-)$ take $x \in N^- \cap f^{-1}(N)$ and assume $f(x) \notin N^-$. Then $f^2(x) \in \operatorname{int} N$, i.e., $f(x) \in f^{-1}(N) \cap N \cap f(N) \subset \operatorname{int} N$ and $x \notin N^-$, a contradiction. This proves property (i). Property (ii) is obvious. To see (iii) observe that $f(\operatorname{Inv} N) = \operatorname{Inv} N \subset \operatorname{int} N$, thus

$$\operatorname{Inv} N \subset \operatorname{int} N \cap f^{-1}(\operatorname{int} N) = \operatorname{int}(N \cap f^{-1}(\operatorname{int} N)) = \operatorname{int}(N \backslash N^-).$$

Finally observe that $\operatorname{cl}(f(N^-) \backslash N) \cap \operatorname{cl}(N \backslash N^-) \subset \operatorname{cl}(f(N) \backslash \operatorname{int} N) \cap \operatorname{cl}(N \cap f^{-1}(\operatorname{int} N))$ $\subset (f(N) \backslash \operatorname{int} N) \cap (N \cap f^{-1}(N)) \subset (f(N) \backslash \operatorname{int} N) \cap (f(N) \cap N \cap f^{-1}(N)) \subset (f(N) \backslash \operatorname{int} N) \cap \operatorname{int} N = \emptyset$. This shows that (N, N^-) is a regular index pair. □

The following proposition is also proved in [E2]. We include the proof for sake of completeness.

Proposition 4.8. *If S is an isolated invariant set then for every neighborhood V of S there exists $N \subset V$ an isolating block for S.*

Proof. By [Mr1, Th. 2] there exist U an open neighborhood of S and continuous functions $\varphi, \gamma : U \to [0, \infty)$ such that

$$\varphi(x) > 0,\ x \in U \cap f^{-1}(U) \qquad \Rightarrow \qquad \varphi(f(x)) < \varphi(x),$$

$$\gamma(x) > 0,\ x \in U \cap f^{-1}(U) \qquad \Rightarrow \qquad \gamma(f(x)) > \gamma(x),$$

$$S \subset \varphi^{-1}(0) \cap \gamma^{-1}(0)$$

and for every neighborhood W of S there exists $\epsilon > 0$ such that

$$M_\epsilon := \operatorname{cl}\{x \in U \mid (\varphi(x) < \epsilon, \gamma(x) < \epsilon\} \subset W.$$

Select $\epsilon > 0$ such that $M_\epsilon \subset U \cap f^{-1}(U) \cap V$. We will show that $M_\epsilon \cap f(M_\epsilon) \cap f^{-1}(M_\epsilon) \subset \operatorname{int} M_\epsilon$. For this end take $x \in M_\epsilon \cap f(M_\epsilon) \cap f^{-1}(M_\epsilon)$. Then $\varphi(x) = \varphi(f(f^{-1}(x))) < \varphi(f^{-1}(x)) \leq \epsilon$ and $\gamma(x) < \gamma(f(x)) \leq \epsilon$. Thus the inequalities hold also in some neighborhood of x and consequently $x \in \operatorname{int} M_\epsilon$. □

Since (N, N^-) is an index pair in N, a corresponding cochain index map and cohomology index map are defined. We will denote them by $f_N^\#$ and f_N^* respectively. The inverse of the inclusion $i^\# : \bar{\Gamma}^*(N \cup f(N^-), N^- \cup f(N^-)) \to \bar{\Gamma}^*(N, N^-)$ will be denoted by λ_N, so that

$$f_N^\# = f^\# \circ \lambda_N$$

$$f_N^* = f^* \circ \lambda_N^*.$$

In the sequel we will also need the following simple proposition.

Proposition 4.9. $f(N \backslash \operatorname{int}_N N^-) \subset N$.

A subset $A \subset S$ is called an attractor (repeller) in S iff there exists a neighborhood U of A in S such that the ω-limit set (α-limit set) of any point $x \in U$ is contained in A. One can verify (see [Mr3]) that if $A^+ \subset S$ is an attractor in S then the set A^- of points in S which have their positive limit set disjoint from A^+ is a repeller in S. The pair (A^-, A^+) is called a repeller-attractor pair in S.

Assume A^-, A^+ are two disjoint invariant subsets of S. The set of connections from A^- to A^+ in S, $\mathrm{Conn}(A^-, A^+; S)$ is defined as the set of those $x \in S$ whose positive limit set is in A^+ and negative limit set is in A^-.

One can easily verify the following proposition.

Proposition 4.10. *If A^-, A^+ are two disjoint invariant subsets of S then (A^-, A^+) is a repeller-attractor in S iff*

$$A^- \cup A^+ \cup \mathrm{Conn}(A^-, A^+; S) = S.$$

$\square$

By Propositions 4.8 and 4.9 every isolated invariant set S admits an index pair (P_1, P_2) in some isolating neighborhood N which isolates S. The pair $(H^*(P_1, P_2), f_P^*)$ is an object in Endo(**G**-GMod), the category of endomorphisms of graded modules over **G** (see [Mr4]). There is a natural functor from this category to its subcategory of automorphisms Auto(**G**-GMod). It assigns to every endomorphism $e : E \to E$ the automorphism induced on the direct limit of the sequence

$$E \xrightarrow{e} E \xrightarrow{e} E \xrightarrow{e} \dots$$

We call it the direct limit functor and denote it by L. It turns out that $L(H^*(P_1, P_2), f_P^*)$ does not depend on the index pair (P_1, P_2) (see [Mr4, Th. 1.7]), hence it is an invariant of N and S. We call it the Conley index of the isolating neighborhood N (or the isolated invariant set S) and denote it $\mathrm{Con}^*(N, f)$ (or respectively $\mathrm{Con}^*(N, S)$). Note that $\mathrm{Con}^*(N, f)$ is a graded module with a distinguished graded automorphism. We will denote the n^{th} component of $\mathrm{Con}^*(N, f)$ by $\mathrm{Con}^n(N, f)$.

Functors other than L can be used in the construction of the Conley index (see [Mr4]). We use L, because of the following useful proposition.

Proposition 4.11. (see [Ma, Th. A.7]). *L preserves exact sequences.*

5. Auxiliary Lemmas.

Lemma 5.1. *Assume N is an isolating block, $N_0, N_1 \subset N$ are compact, $N = N_0 \cup N_1$ and $N_0 \cap N_1 = \emptyset$. Then N_0 and N_1 are isolating blocks. Moreover if $E_0 := N^- \cup N_1$, $E_1 := N^- \cup N_0$ then*

$$\bar{\Gamma}^*(N, N^-) = \bar{\Gamma}^*(N, E_0) \oplus \bar{\Gamma}^*(N, E_1) \tag{9}$$

$$\bar{H}^*(N, N^-) = \bar{H}^*(N, E_0) \oplus \bar{H}^*(N, E_1).$$

Furthermore, if

$$f_N^* = \begin{bmatrix} f_N^*(0,0) & f_N^*(0,1) \\ f_N^*(1,0) & f_N^*(1,1) \end{bmatrix} \tag{9'}$$

denotes the corresponding decomposition of the index map f_N^* into partial cochain maps, then for $i = 0, 1$

$$\mathrm{Con}(N_i, f) = L(H^*(N, E_i), f_N^*(i,i)). \tag{10}$$

Finally, if $f_N^*(i)$ denotes the map f_N^* with the $(1-i)^{th}$ row in (9') replaced with zero then also

$$\mathrm{Con}(N_i, f) = L(H^*(N, N^-), f_N^*(i)). \tag{11}$$

Proof. N_0 and N_1 are isolating blocks by Proposition 4.6. Formula (9) follows immediately from Proposition 3.5. Since proofs of (10) and (11) for $i = 0, 1$ are analogous, without loss of generality we can assume that $i = 0$. Put $g := f_N^{\#}(0,0)$ and

$$E_{00} := E_0 \cup (N_0 \cap f^{-1}(N_1)), \quad E_{01} := E_0 \cup (N_0 \cap f^{-1}(N_0)).$$

Observe that

$$N_0 = (N_0 \backslash f^{-1}(N_0)) \cup (N_0 \backslash f^{-1}(N_1))$$

and

$$N_0 \backslash f^{-1}(N_i) \subset N^- \cup (N_0 \cap f^{-1}(N_{1-i})), \quad i = 0, 1.$$

Hence

$$\begin{aligned} N =& N_1 \cup N_0 = N_1 \cup (N_0 \backslash f^{-1}(N_0)) \cup N_1 \cup (N_0 \backslash f^{-1}(N_1)) \\ &\subset \operatorname{int}_N(N^- \cup N_1 \cup (N_0 \cap f^{-1}(N_1))) \cup \operatorname{int}_N(N^- \cup N_1 \cup (N_0 \cap f^{-1}(N_0))) \\ &= \operatorname{int}_N E_{00} \cup \operatorname{int}_N E_{01} \subset N, \end{aligned}$$

and we obtain from Proposition 3.5 that

$$\bar{\Gamma}^*(N, E_0) = \bar{\Gamma}^*(N, E_{00}) \oplus \bar{\Gamma}^*(N, E_{01}).$$

We claim that the corresponding decomposition of g is

$$g = \begin{bmatrix} g(0,0) & g(0,1) \\ g(1,0) & g(1,1) \end{bmatrix} = \begin{bmatrix} g(0,0) & g(0,1) \\ 0 & 0 \end{bmatrix}, \tag{12}$$

i.e. the bottom row in the decomposition is zero.

To see this take $[\varphi] \in \bar{\Gamma}^*(N, E_0)$. By Proposition 3.5 we have to show that

$$\varphi' := \left(\kappa_{E_{01}} \circ \kappa_{E_0} \circ f^{\#} \circ \lambda_N \circ i^{\#}_{E_0}\right)(\varphi)$$

is locally zero for all $\varphi \in \Gamma^*(N, E_0)$.

¿From the definitions of the maps $\kappa_{E_{01}}, \kappa_{E_0}, \lambda_N$ it follows that

$$\varphi'(x_0, \ldots, x_p) = \begin{cases} \varphi(f(x_0), \ldots, f(x_p)) & \text{if } x_0, \ldots, x_p \in E_{00} \cap E_1 \cap f^{-1}(N) \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Take $x \in N$. Since $\varphi' \in \Gamma^*(N, E_{01})$, φ' is zero in some neighborhood of every point in $\operatorname{int}_N E_{01}$. If $x \in f^{-1}(\operatorname{int}_N E_0)$ then we can take a neighborhood U of $f(x)$ such that $U \subset E_0$ and $\varphi|_U = 0$. Hence if V is a neighborhood of x such that $f(V) \subset U$ then $\varphi'|_V = 0$.

Thus, it remains to consider the case $x \in N \backslash (\operatorname{int}_N E_{01} \cup f^{-1}(\operatorname{int}_N E_0))$. In particular we have then $x \in N_0$, $x \notin \operatorname{int}_N N^-$, $f(x) \notin \operatorname{int}_N N^-$, $f(x) \notin N_1$. By Proposition 4.9 $f(x) \in N$, thus $f(x) \in N_0$. We will show that $x \notin N^-$. If $x \in N^-$ then $x \in N \backslash \operatorname{int}_N N^-$ and by Proposition 4.9 $f(x) \in N$. Applying it again we find that also $f^2(x) \in N$. Thus $f(x) \in f^{-1}(N) \cap N \cap f(N) \subset \operatorname{int} N$. It follows that $x \notin N^-$, a contradiction. Hence $x \notin N^-$ and since $x \in N_0$ we see that $x \notin E_0$. Take V a neighborhood of x such that

$V \cap E_0 = \emptyset$ and $f(V) \subset N_0$. Let $x_0, \dots, x_p \in V$. For every $i = 0, 1, \dots, p$ there is $x_i \notin E_{00}$ because otherwise we get from $x_i \notin E_0$ that $f(x_i) \in N_1$. It follows now from (13) that $\varphi'|_V = 0$ which proves that φ' is locally zero.

Now observe that

$$N_0^- = N_0 \backslash f^{-1}(\operatorname{int} N \backslash N_1) = (N_0 \backslash f^{-1}(\operatorname{int} N)) \cup (N_0 \cap f^{-1}(N_1)) = N_0 \cap E_{00}.$$

Hence the inclusion $i_0 : (N_0, N_0^-) \hookrightarrow (N, E_{00})$ is an excision of N_1 and since $\operatorname{cl}_N N_1 = N_1 = \operatorname{int}_N N_1 \subset \operatorname{int}_N E_{00}$, it induces an isomorphism $i_0^\# : \bar{\Gamma}^*(N, E_{00}) \to \bar{\Gamma}^*(N_0, N_0^-)$ (see [Sp, p. 309]). We will show that the following diagram

$$\text{(14)} \qquad \begin{array}{ccccc} \bar{\Gamma}^*(N, E_{00}) & & \xrightarrow{\quad g(0,0) \quad} & & \bar{\Gamma}^*(N, E_{00}) \\ \downarrow{\scriptstyle i_0^\#} & & & & \downarrow{\scriptstyle i_0^\#} \\ \bar{\Gamma}^*(N_0, N_0^-) & \xrightarrow{\lambda_{N_0}} & \bar{\Gamma}^*(N_0 \cup f(N_0^-), N_0^- \cup f(N_0^-)) & \xrightarrow{f^\#} & \bar{\Gamma}^*(N_0, N_0^-) \end{array}$$

commutes.

To this end take $[\varphi] \in \Gamma^p(N, E_{00})$ and set

$$\psi_1 := (i_0^\# \circ g(0,0))(\varphi), \quad \psi_2 := (f^\# \circ \lambda_{N_0} \circ i_0^\#)(\varphi).$$

Observe that $\psi_1 = (i_0^\# \circ \kappa_{E_{00}} \circ \kappa_{E_0} \circ f^\# \circ \lambda_N \circ i_{E_0}^\# \circ i_{E_{00}}^\#)(\varphi)$. Thus a straightforward computation shows that

$$\psi_1(x_0, \dots, x_p) = \begin{cases} \varphi(f(x_0), \dots, f(x_p)) & \text{if } x_0, \dots, x_p \in N_0 \cap E_{01} \cap f^{-1}(N) \\ 0 & \text{otherwise.} \end{cases}$$

Similarly we find that

$$\psi_2(x_0, \dots, x_p) = \begin{cases} \varphi(f(x_0), \dots, f(x_p)) & \text{if } x_0, \dots, x_p \in N_0 \cap f^{-1}(N_0) \\ 0 & \text{otherwise.} \end{cases}$$

We need to show that $\psi := \psi_2 - \psi_1$ is locally zero. Thus take $x \in N_0$. Obviously ψ disappears around points $x \in \operatorname{int}_{N_0} N_0^-$, because $\psi \in \Gamma^p(N_0, N_0^-)$. Thus we can assume that $x \notin \operatorname{int}_{N_0} N_0^-$ and similarly that $f(x) \notin \operatorname{int}_N E_{00}$. It follows from Proposition 4.9 that $f(x) \in N_0$. In particular $f(x) \notin N_1$. Choose V a neighborhood of x_0 in N_0 such that $f(V) \cap N_1 = \emptyset$ and take $x_0, \dots, x_p \in V$. If for some $i = 0, \dots p$ $f(x_i) \notin N$ then $\psi_1(x_0, \dots, x_p) = \psi_2(x_0, \dots, x_p) = 0$, so $\psi(x_0, \dots, x_p) = 0$. Otherwise $f(x_i) \in N_0$ for all $i = 0, \dots, p$. In particular $x_0, \dots, x_p \in N_0 \cap f^{-1}(N_0) \subset N_0 \cap E_{01} \cap f^{-1}(N)$ and $\psi_1(x_0, \dots, x_p) = \psi_2(x_0, \dots, x_p) = \varphi(f(x_0), \dots, f(x_p))$, i.e., $\psi(x_0, \dots, x_p) = 0$. This shows that $\psi|_V = 0$ and completes the proof of the commutativity of diagram (14).

It follows from Proposition 3.5 and the structure of g^* given by (12) that the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(N, E_{00}) & \xrightarrow{\iota^*_{E_{00}}} & H^*(N, E_0) & \xrightarrow{\kappa^*_{E_{01}}} & H^*(N, E_{01}) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle g^*(0,0)} & & \Big\downarrow{\scriptstyle g^* = f^*_N(0,0)} & & \Big\downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & H^*(N, E_{00}) & \longrightarrow & H^*(N, E_0) & \longrightarrow & H^*(N, E_{01}) & \longrightarrow & 0 \end{array} \tag{15}$$

commutes. Proposition 4.11 implies that the sequence

$$0 \longrightarrow L(H^*(N, E_{00}), g^*(0,0)) \longrightarrow L(H^*(N, E_0), g^*) \longrightarrow L(H^*(N, E_{01}), 0) \longrightarrow 0$$

is exact. However obviously $L(H^*(N, E_{01}), 0) = 0$, thus $L(H^*(N, E_{00}), g^*(0,0))$ and $L(H^*(N, E_0), g^*)$ are isomorphic. Now, applying the direct limit functor to diagram (14) we conclude that $L(H^*(N, E_{00}), g^*(0,0))$ and $L(H^*(N_0, N_0^-), f^*_{N_0})$ are isomorphic. Since $L(H^*(N_0, N_0^-), f^*_{N_0}) = \mathrm{Con}(N_0)$ by definition, the proof of (10) is finished.

Finally an argument analogous to that used with diagram (15) shows that also (11) is satisfied. $\square$

Proposition 5.2. *Assume $F : V \to V$ is a linear endomorphism of a $\mathbf{G}$-module V and $L(V, F) = (\mathbf{G}, \mathrm{id})$. Then 1 is an eigenvalue of F and for every $v \in V$ there exists $k \in \mathbf{N}$ such that either $F^k v = 0$ or $F^k v$ is a 1–eigenvector of F. In particular the generalized 1–eigenspace of F is one-dimensional.*

Proof. The definition of the direct limit implies that we have a commutative diagram

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[dr, "p_0"'] & V \arrow[r, "F"] \arrow[d, "p_1"] & V \arrow[dl, "p_2"] & \cdots \\ & \mathbf{G} & & \end{tikzcd}$$

and since F induces the identity on the limit we have also a family of commutative diagrams

$$\begin{array}{ccc} V & \xrightarrow{F} & V \\ \Big\downarrow{\scriptstyle p_i} & & \Big\downarrow{\scriptstyle p_i} \\ \mathbf{G} & \xrightarrow{\mathrm{id}} & \mathbf{G}. \end{array} \tag{16}$$

It follows from [Ma, Prop. A.4] that

$$\operatorname{gker} F := \bigcup_{n \geq 0} \ker F^n = \ker p_i \text{ for all } i \in \mathbf{N} \tag{17}$$

and p_i is surjective for some $i \in \mathbf{Z}$. In particular we get $p_i = p_0$ for all $i \in \mathbf{N}$. Select $w \in V$ such that $p_0(w) \neq 0$. It follows from (16) that $p_0(w - Fw) = 0$. Hence (17) implies $F^n(w - Fw) = 0$ for some $n \in \mathbf{N}$, i.e. $Fz = z$ for $z := F^n w$. Since $w \notin \ker p_0$, we see from (17) that $z \neq 0$. This shows that 1 is an eigenvalue of F and z is a corresponding eigenvector. We get from (16) that for any $v \in V$ $p_0(v - Fv) = 0$. It follows from (17) that for some $n \in \mathbf{N}$, $F^n(v - Fv) = 0$. Hence either $F^n v = 0$ or $F^n v$ is a 1–eigenvector of F.$\square$

6. Proofs of main results

Theorem 2.1 follows immediately from Propositions 4.1 and 4.6.

Proof of Theorem 2.2. By Proposition 4.8 one can assume that $N_{k,l} = N_k \cup N_l$ is an isolating block.

(i) The first step is to show that

$$N_k \cap S_{\{kk,kl,ll\}} \subset \operatorname{Inv}^+(N_k). \tag{18}$$

To this end take $x \in N_k \cap S_{\{kk,kl,ll\}}$. By definition

$$S_{\{kk,kl,ll\}} = \operatorname{Inv}\big((N_k \cap f(N_k)) \cup (N_k \cap f(N_l)) \cup (N_l \cap f(N_l))\big)$$

hence,

$$\begin{aligned} x &\in N_k \cap \operatorname{Inv}\big((N_k \cap f(N_k)) \cup (N_k \cap f(N_l))\big) \\ &\subset N_k \cap \operatorname{Inv}(N_k). \end{aligned}$$

Thus $f(x) \in N_k$. A similar argument proves that

$$f(N_l) \cap S_{\{kk,kl,ll\}} \subset \operatorname{Inv}^-(N_l). \tag{19}$$

Observe that $S_j = \operatorname{Inv}(N_j) = \operatorname{Inv}(N_j \cap f(N_j)) \subset S_{\{kk,kl,ll\}}$ for $j = k, l$. Hence

$$S_l \cup S_k \cup \operatorname{Conn}(S_l, S_k; S_{\{kk,kl,ll\}}) \subset S_{\{kk,kl,ll\}}.$$

The opposite inclusion follows immediately from (18) and (19). Hence Proposition 4.10 shows that (S_k, S_l) is a attractor-repeller pair in $S_{\{kk,kl,ll\}}$.

(ii) To simplify the notation let

$$M = N_{\{kk,kl,lk,ll\}}$$

and let

$$E = M^- := M \setminus f^{-1}(M).$$

The cohomology indices of interest for our computation are:

$$
\begin{aligned}
V &= \bar{H}^*(M, E),\\
V_k &= \bar{H}^*(M, E \cup N_{ll} \cup N_{lk}),\\
V_l &= \bar{H}^*(M, E \cup N_{kk} \cup N_{kl}),\\
V_{kk} &= \bar{H}^*(M, E \cup N_{ll} \cup N_{lk} \cup N_{kl}),\\
V_{ll} &= \bar{H}^*(M, E \cup N_{kk} \cup N_{lk} \cup N_{kl}),\\
V_{kl} &= \bar{H}^*(M, E \cup N_{ll} \cup N_{lk} \cup N_{kk}),\\
V_{lk} &= \bar{H}^*(M, E \cup N_{ll} \cup N_{kk} \cup N_{kl}).
\end{aligned}
$$

Lemma 5.1 applied to M, $N_{kl} \cup N_{kk}$, and $N_{ll} \cup N_{lk}$ implies that

$$V = V_k \oplus V_l.$$

Similarly, for $i = k, l$,

$$V_i = V_{ik} \oplus V_{il}.$$

$f_M^* : V \to V$ can be decomposed as

$$f_M^* = \begin{bmatrix} F_{kk} & F_{kl} \\ F_{lk} & F_{ll} \end{bmatrix} \qquad \text{where } F_{ij} : V_j \to V_i \quad i, j \in \{k, l\} \tag{20}$$

and each F_{ij} can be further decomposed as

$$F_{ij} = \begin{bmatrix} B_{ikjk} & B_{ikjl} \\ B_{iljk} & B_{iljl} \end{bmatrix} \text{where } B_{imjn} : V_{jn} \to V_{im}, \quad m, n \in \{k, l\}. \tag{21}$$

Since $f(N_{ij}) \subset N_{ki} \cup N_{li}$, one can use the argument of Lemma 5.1 which gave rise to (12) to conclude that

$$f_M^* = \begin{bmatrix} B_{kkkk} & B_{kkkl} & 0 & 0 \\ 0 & 0 & B_{kllk} & B_{klll} \\ B_{lkkk} & B_{lkkl} & 0 & 0 \\ 0 & 0 & B_{lllk} & B_{llll} \end{bmatrix}.$$

Applying (20) and Lemma 5.1 to $M, N_{kl} \cup N_{kk}, N_{lk} \cup N_{ll}$ results in

$$\operatorname{Con}(N_k) = L\left(V, \begin{bmatrix} F_{kk} & F_{kl} \\ 0 & 0 \end{bmatrix}\right) = L(V_k, F_{kk}) \tag{22}$$

and

$$\mathrm{Con}(N_l) = L\left(V, \begin{bmatrix} 0 & 0 \\ F_{lk} & F_{ll} \end{bmatrix}\right) = L(V_l, F_{ll}) \tag{23}$$

Applying (21) and Lemma 5.1 to $M, N_{lk}, N_{kk} \cup N_{kl} \cup N_{ll}$ we find that

$$\mathrm{Con}(N_{kk} \cup N_{kl} \cup N_{ll}) = L(V, \bar{F}_{kl}) \quad \text{with} \quad \bar{F}_{kl} = \begin{bmatrix} F_{kk} & F_{kl} \\ 0 & F_{ll} \end{bmatrix} \tag{24}$$

Let $\iota : V_k \to V$ and $\pi : V \to V_l$ denote the inclusion and projection corresponding to the direct sum $V = V_k \oplus V_l$. Then, we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_k & \xrightarrow{\iota} & V & \xrightarrow{\pi} & V_l & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F_{kk}} & & \downarrow{\scriptstyle \bar{F}_{kl}} & & \downarrow{\scriptstyle F_{ll}} & & \\ 0 & \longrightarrow & V_k & \xrightarrow{\iota} & V & \xrightarrow{\pi} & V_l & \longrightarrow & 0 \end{array} \tag{25.}$$

Applying the functor L and using the assumptions (22), (23) and (24) gives the commutative diagram (1). Since, by Proposition 4.11 L preserves exact sequences, the rows in (1) are exact.

(iii) follows immediately from (1) and (iv) is a consequence of the additivity property of the Conley index (see [Mr4]). □

Proof of Theorem 2.3. Let m be as in (4). We will follow the notation introduced in the proof of Theorem 2.2 under the assumption that all graded endomorphisms $F_{ij}, \bar{F}_{ij}$ introduced there are now restricted to the m^{th} level. Observe that by (4) and Proposition 5.2 F_{ii} has a unique (up to a constant) 1–eigenvector v_i. We will show that $F_{ij}v_j \notin \operatorname{gker} F_{ii}$ (comp (17)). If $F_{ij}v_j \in$ gker F_{ii} then $F_{ii}^p F_{ij} v_j = 0$ for some natural number q. Put

$$w_0 = \left(v_j, \sum_{n=0}^{q-1} F_{ii}^n F_{ij} v_j\right), \qquad w_1 = (0, v_i)$$

It is straightforward to verify that w_0 and w_1 are linearly independent 1–eigenvectors of $\bar{F}_{ij}$. Also, an easy computation shows that $D_{ij} = L(\bar{F}_{ij})$ has two linearly independent 1–eigenvectors. This implies that $\alpha_{ij} = 0$, a contradiction. Hence $F_{ij}v_j$ is not in the generalized kernel of F_{ii}. By Proposition 5.2 there exists a natural number d_{ij} such that $F_{ii}^{d_{ij}} F_{ij} v_j$ is a 1–eigenvector of F_{ii}.

Take $d := 1 + \max_{1 \le i,j \le K}\{d_{i,j}\}$. Then

$$F_{ii}^{d-1} F_{ij} v_j \text{ is a 1–eigenvector of } F_{ii} \text{ for any } i, j \in \{1, \dots, K\}, i \neq j. \tag{26}$$

Define the map $\beta : N \to \{1, \dots, K\}$ by

$$\beta(x) := k \text{ if } x \in N_k$$

and the map $p : \text{Inv } N \to \Sigma_K$ by

$$p(x) := (\beta(f^{-nd}(x)))_{n \in \mathbf{Z}}$$

Obviously β and p are continuous (observe that this is the point at which the assumption that $\alpha_{kl} = 0$ implies $\text{Conn}(S_l, S_k) = \emptyset$ is used) and $p \circ f^d = \sigma \circ p$. It remains to show that p is surjective. For this end take an element $\alpha : \mathbf{Z} \to \{1, \dots, K\}$ of $\Sigma_{\mathcal{A}}$ and fix $n \in \mathbf{N}$. Let $\varphi_i \in v_i$ be a cochain representative of v_i. Define recursively a sequence

$$\begin{aligned} \psi_{-n} &:= \varphi_{\alpha_{-n}} \\ \psi_i &:= [f_M^{\#,k}(\alpha_i, \alpha_i)^{d-1} \circ f_M^{\#,k}(\alpha_i, \alpha_{i-1})](\psi_{i-1}) \quad i = -n+1, -n+2, \dots, n. \end{aligned} \tag{27}$$

Then $\psi_i \in \bar{\Gamma}^k(N, E_{\alpha_i})$. In particular $|\psi_i| \subset \text{cl}(N \backslash E_{\alpha_i}) \subset N_{\alpha_i}$. From (27) and Proposition 3.1 one has that

$$f^j(|\psi_i|) \subset N, \qquad j = 0, 1, \dots d$$

and

$$f^d(|\psi_i|) \subset |\psi_{i-1}| \qquad i = -n+1, -n+2, \dots, n,$$

which implies that

$$f^j(|\psi_n|) \subset N, \qquad j = 0, 1, \dots 2nd \tag{28}$$

and

$$f^{md}(|\psi_n|) \subset N_{\alpha_{n-m}} \qquad m = 0, 1, \dots, 2n. \tag{29}$$

By (26) and (27) the cohomology class of ψ_n is a 1–eigenvector of $F_{\alpha_n \alpha_n} = f_M^k(\alpha_n, \alpha_n)$. Hence this class is different from zero and by Proposition 3.2 $|\psi_n| \neq \emptyset$. Select $y_n \in |\psi_n|$ and put $x_n := f^{nd}(y_n)$. Then, by (28)

$$f^i(x_n) \in N, \qquad i = -nd, -nd+1, \dots nd \tag{30}$$

and by (29)

$$(31) \qquad f^{-id}(x_n) = f^{(n-i)d}(y_n) \in f^{(n-i)d}(|\psi_n|) \subset N_{\alpha_i} \qquad i = -n, -n+1, \dots, n.$$

In particular $x_n \in N_{\alpha_0}$. Since N_{α_0} is compact, one can assume that $x_n \to x_* \in N_{\alpha_0}$. From (30) one obtains $x_* \in \operatorname{Inv} N$ and from (31)

$$f^{-id}(x_*) \in N_{\alpha_i} \qquad \text{for } i \in \mathbf{Z},$$

i.e. $\beta(f^{-id}(x_*)) = \alpha_i$ for $i \in \mathbf{Z}$. This shows that $p(x_*) = \alpha$ and proves that p is surjective.
□

Proof of Theorem 2.4. Define the map p as in the proof of Theorem 2.3. Let $\alpha : \mathbf{Z} \to \{0,1\}$ be a periodic sequence in Σ_2 with period q. By Theorem 2.1 M_α is an isolating neighborhood for all f_t, ($t \in [0,1]$). In particular $(f_t)^q$ has no fixed points in the boundary of M_α. Thus, the fixed point index $i((f_t)^q, \operatorname{int} M_\alpha)$ is defined (see [Do,7.5]) and

$$i((f_0)^q, \operatorname{int} M_\alpha) = i((f_1)^q, \operatorname{int} M_\alpha).$$

The multiplicativity property of the fixed point index implies that

$$i((f_0)^q, \operatorname{int} M_\alpha) = i((f_0|_{\mathbf{R}^2}, \operatorname{int} N_\alpha).$$

The latter is the fixed point index of a hyperbolic fixed point, hence it is different from zero. Thus there exists $x \in \operatorname{Inv} M$, a q-periodic point of f such that $p(x) = \alpha$. The surjectivity of p follows from the fact that periodic sequences are dense in Σ_2 and $\operatorname{Inv} N$ is compact. □

7. The Henon Map

Recall that the Henon map is given by

$$H(x,y) = (1 + y - ax^2, bx).$$

In order to provide several examples of how our ideas can be applied to specific systems we shall study the dynamics of H for the set of parameter values

$$\{(a,b) \mid a = 1.8,\ 0 \leq b \leq 0.025\}.$$

Simple numerical experiments suggest that for these parameter values the Henon map has an attractor, upon which the dynamics may be rather complicated. The first observation which must be made is that Theorem 2.3 cannot provide direct information about the entire structure of an attractor. In order for $\alpha_{kl} = 1$, it must be that $\mathrm{Inv}(N_l)$ have an unstable "manifold". This forces the $\mathrm{Inv}(N)$ to be unstable. What is needed is a theorem, based on the index information, which can provide information concerning the structure of a chaotic attractor. How to do this in an applicable manner remains an open question.

We avoid this problem by studying the invariant set generated by H^2, the second iterate of H, of a neighborhood $\widetilde{N}$ which we shall defined below. Observe that proving the existence of a semi-conjugacy from $\mathrm{Inv}(\widetilde{N}, H^2)$ to a horseshoe, provides a lower bound on the complexity of the dynamics of H.

Before stating our result we introduce some notation which will be used throughout this section. Let

$$\begin{array}{ll} p_1 = -0.54198 & p_2 = -0.403384 \\ p_3 = -0.348221 & p_4 = -0.178249 \\ p_5 = 0.178249 & p_6 = 0.348221 \\ p_7 = 0.403384 & p_8 = 0.54198 \end{array}$$

and set

$$N = \bigcup_{i=1}^{4} [p_{2i-1}, p_{2i}].$$

We shall make use of the following functions:

$$\begin{aligned}
h_1(x,y) &:= 1 + bx - a(1 + y - ax^2)^2 \\
h_2(x,y) &:= b(1 + y - ax^2) \\
h_1^{-1}(x,y) &:= \frac{x-1}{b} + \frac{ay^2}{b^3} \\
h_2^{-1}(x,y) &:= \frac{y}{b} - 1 + \frac{a}{b^6}(b^2(x-1) + ay^2)^2 \\
q(b) &:= \frac{b(1 - ap_5^2)}{1-b}.
\end{aligned}$$

Observe that

$$H^2(x,y) = (h_1(x,y), h_2(x,y))$$

and

$$H^{-2}(x,y) = (h_1^{-1}(x,y), h_2^{-1}(x,y)).$$

Theorem 7.1. *Assume the parameter values a, $b > 0$ lie in a sufficiently small open neighborhood of $\{(a,b) \mid a = 1.8,\ 0 < b \leq 0.025\}$ and that*

$$\widetilde{N} := N \times [0, q(b) + \mu]$$

for some $\mu > 0$, but arbitrarily small. Then, there exists a semi-conjugacy from $\mathrm{Inv}(\widetilde{N}, H^2)$ onto the full shift dynamics on two symbols, i.e. the invariant set of the horseshoe.

The proof of this Theorem occupies the rest of this section and will be done in three stages each of which is meant to describe a method for applying the Conley index theory to maps. As was indicated in the introduction, this result should be viewed as an example, included for the purposes of demonstrating the applicability of our abstract theorems. It is not intended to be a careful analysis of the Henon map.

The first stage of the proof involves a singular perturbation technique. It is well known that for $b = 0$, the invariant set of H is given by the invariant set of a 1-dimensional map. Using this map we find isolating neighborhoods and compute the Conley indices. We then show that the index information for the 1-dimensional map lifts to the two dimensional map which in turn implies that the index computations are valid for $0 \leq b \leq \epsilon$. This lifting of one dimensional information to two dimensions justifies calling this a singular perturbation technique.

The second stage is referred to as a perturbation with *a priori* bounds; the estimates of the one dimensional map are used to determine an interval of parameter values, $a = 1.8$, $0 \leq b \leq 0.015$ on which the index computations remain valid. Observe that this is a significantly stronger result since we have a lower bound on the set of parameter values for which the theorem is valid. The importance of this with regards to modeling of physical phenomenon is obvious.

The third stage uses the continuation property of the index. The goal is to see for what parameter values the isolating neighborhood $\widetilde{N}$ remains an isolating block. In our case this is shown to be hold over the interval $a = 1.8$, $0.015 \leq b \leq 0.025$.

7.A. Singular Perturbation.

As was mentioned above we begin our proof by considering H^2 with $b = 0$, i.e.

$$H^2(x,y) = (1 - a(1 + y - ax^2)^2, 0).$$

This implies that

$$\mathrm{Inv}(N \times [-y_0, y_0], H^2) \subset \mathbf{R} \times \{0\}$$

which allows us to restrict our attention to the one dimensional map

$$F(x) = 1 - a(1 - ax^2)^2$$

the graph of which is shown in Figure 3

We shall now show that, aside from the fact that F is not a homeomorphism, all the hypothesis of Theorem 2.3 are satisfied. The dynamics of maps of the form of F are well understood (see [De]) thus we leave the details of the following argument to the reader. Let $N_1 = [p_1, p_4]$ and $N_2 = [p_5, p_8]$. A simple calculation shows that for $a = 1.8$, $F(p_i) \notin N_1 \cup N_2$ for $i = 1, 4, 5, 8$, and hence, $N_1 \cup N_2$ is an isolating neighborhood. Because F is one dimensional and monotone over the sets N_1 and N_2, it is easy to check that we can choose

$$\begin{array}{ll} N_{11} = [p_3, p_4] & N_{22} = [p_7, p_8] \\ N_{12} = [p_5, p_6] & N_{21} = [p_1, p_2] \end{array}.$$

Again the monotonicity of F over N_1 and N_2, makes the calculations that

$$\mathrm{Con}^1(\mathrm{Inv}(N_k, F), \mathbf{Z}_2) \approx (\mathbf{Z}_2, id) \qquad k = 1, 2,$$

and that

$$\alpha_{kl} = 1 \qquad k, l = 1, 2,$$

straightforward.

Figure 3. Graph of $F(x) = 1 - 1.8(1 - 1.8x^2)^2$.

Proposition 7.2. *Given $y_0 > 0$ and $a = 1.8$, there exists $\epsilon > 0$ such that for all $b \in [0, \epsilon]$, $N \times [-y_0, y_0]$ is an isolating neighborhood under H^2. Furthermore*

$$\mathrm{Con}^1(\mathrm{Inv}(N_k \times [-y_0, y_0], H^2), \mathbf{Z}_2) \approx (\mathbf{Z}_2, id) \qquad k = 1, 2$$

and

$$\alpha_{kl} = 1 \qquad k, l = 1, 2,$$

where the four smaller isolating neighborhoods are given by

$$N_{11} = [p_3, p_4] \times [-y_0, y_0] \qquad N_{22} = [p_7, p_8] \times [-y_0, y_0]$$

$$N_{12} = [p_5, p_6] \times [-y_0, y_0] \qquad N_{21} = [p_1, p_2] \times [-y_0, y_0].$$

Proof: N is an isolating neighborhood for F and for $b = 0$, $\mathrm{Inv}(N \times [-y_0, y_0], H^2) \subset \mathbf{R} \times \{0\}$, therefore given $y_0 > 0$, there exists $\epsilon > 0$ such that for all $b \in [0, \epsilon]$, $N \times [-y_0, y_0]$ is an isolating neighborhood under H^2. This last comment is the standard local continuation property of the Conley index. Now, the commutativity property of the Conley index (see [Mr4]) comes into play to guarantee that

$$\mathrm{Con}^1(\mathrm{Inv}(N_{kl}\times[-y_0,y_0],H^2),\mathbf{Z}_2)\approx Con^1(\mathrm{Inv}(N_{kl},F),\mathbf{Z}_2)\quad k,l=1,2.$$

For the same reason, the α_{kl}'s remain the same. $\square$

We can apply Theorem 2.3 for $0<b<\epsilon$. However, this only provides a semi-conjugacy of a power of H^2 with the shift dynamics. In theorem 7.1 we claim that the power can be chosen to be 1. To see this one merely applies the proof of Theorem 2.4 at this point.

Lemma 7.3. *For $a=1.8$ and $0<b<\epsilon$,*

$$\mathrm{Inv}(N\times[-y_0,y_0],H^2)=\mathrm{Inv}(N\times[0,y_0],H^2)$$

Proof: It is sufficient to observe that for $x\in[p_1,p_8]$ and $|y|\le y_0$ for y_0 sufficiently small, $1+y-1.8x^2>0$. $\square$

7.B. Perturbation with A priori Bounds.

In this case we shall present a fairly simple analytic technique for showing that Theorem 7.1 is valid on a fixed set of parameter values. The idea of the proof is intuitively quite simple: since our isolating neighborhood $\widetilde{N}$ consists of rectangles, for b close to zero the image of $\widetilde{N}$ under H^2 should be made up of flattened rectangles whose bases are determined by F acting on $N\times\{0\}$. What needs to be shown is that the invariant set in $\widetilde{N}$ does not intersect the *top*,

$$N\times\{q(b)\},$$

bottom,

$$N\times\{0\},$$

or *sides*

$$\sigma_i=p_i\times[0,q(b)]\qquad i=1,\dots,8$$

of the rectangles.

This is done in the following three lemmas.

Lemma 7.4. *If $x \in N$, $a = 1.8$ and $b > 0$, then*

$$(x, 0) \notin \mathrm{Inv}(\widetilde{N}, H^2).$$

Proof: $h_2(x, y) > 0$ for all $y \geq 0$. $\square$

Lemma 7.5. *If $x \in N$, $a = 1.8$ and $b > 0$, then*

$$(x, q(b)) \notin \mathrm{Inv}(\widetilde{N}, H^2).$$

Proof: We shall show that for all (x, y) such that $x \in N$ and $0 \leq y \leq q(b)$, $h_2(x, y) \leq q(b)$.

$$\begin{aligned} h_2(x, y) &= b(1 + y - 1.8x^2) \\ &\leq b(1 + q(b) - 1.8p_5^2) \\ &\leq q(b) \end{aligned}$$

It is important to observe that the only points at which one obtains equality are $(p_4, q(b))$ and $(p_5, q(b))$. In the next lemma it will be shown that for the prescribed values of a and b, $h_1(p_4, q(b)), h_1(p_5, q(b)) \notin N$. $\square$

Lemma 7.6. *For $a = 1.8$ and $0 < b \leq 0.015$,*

$$h_1(\sigma_i) \cap N = \emptyset.$$

Proof: Define

$$G(x, y) := bx - ay^2 - 2ay + 2a^2yx^2.$$

Then $h_1(x, y) = F(x) + G(x, y)$. Let $\delta = 0.05802$. We leave it to the reader to check that

$$\delta < \min\{|f(p_k) - p_l| \mid k, l = 1, \ldots, 8\}.$$

We shall now show that for $(x, y) \in N \times [0, q(b)]$,

$$|G(x, y)| < \delta.$$

Observe that since we know that the endpoints of N under F are mapped outside of N, this finishes the proof.

Our first claim is that the maxima and minima of $G(x,y)$ restricted to $\widetilde{N}$ occur on the boundary. To see this observe that

$$\frac{\partial G}{\partial y} = 2a(-1 - y + ax^2).$$

Hence, $\frac{\partial G}{\partial y} < 0$ for $(x,y) \in \widetilde{N}$. Therefore, the maximum of G on $\widetilde{N}$ occurs at a point of the form $(x,0)$ and the minimum occurs at a point of the form $(x, q(b))$. Now $G(x,0) = bx$ hence the maximum occurs at $(p_8, 0)$. But for $0 < b < 0.01$, $|G(p_8,0)| < \delta$. From $\frac{\partial G}{\partial x}(x, q(b))$ it is easy to determine that the minimum value of G occurs at $(p_4, q(b))$ and hence it is straightforward to check that $|G(p_4, q(b))| < \delta$ for the desired range of b. $\square$

By continuation of the Conley index, the indexes of the various isolating neighborhoods remain constant for all $0 < b \leq 0.015$ and hence the results of Proposition 7.2 can be extended to $b = 0.015$. It is worth mentioning that $|G(p_4, q(0.025))| \not< \delta$, thus we need a different technique to complete the proof of Theorem 7.1

7.C. Continuation

The results of 7.B. were obtained by using the estimates for $b = 0$. These estimates are not sufficient in the range of values $0.015 \leq b \leq 0.025$ which we shall now study. In some sense, this is the region of parameter values at which the results become interesting, i.e. the estimates which determine the dynamics at these parameter values must be done in the full two dimensional system. Lemmas 7.4 and 7.5 are valid for all parameter values, however, we can no longer assume that $h_1(\sigma_i) \cap N = \emptyset$. Thus, we need to study the images of the sets σ_i, $i = 1, \ldots, 8$.

The following observation is trivial but crucial. If $(x,y) \in \mathrm{Inv}(\widetilde{N})$, then:

$$p_{2i-1} \leq h_1(x,y) \leq p_{2i} \qquad i = 1,2,3,4 \tag{32}$$

$$p_{2i-1} \leq h_1^{-1}(x,y) \leq p_{2i} \qquad i = 1,2,3,4. \tag{33}$$

We shall make use of the following functions;

$$g(x,p,b) = ax^2 - 1 + \sqrt{\frac{1 + bx - p}{a}}$$

$$f(x,p,b) = b\sqrt{\frac{bp + 1 - x}{a}}.$$

Lemma 7.7. *If* $(x,y) \in \mathrm{Inv}(\widetilde{N})$, *then*

$$\left.\begin{array}{r} 0 \leq y \leq g(x, p_{2i-1}, b) \\ g(x, p_{2i}, b) \leq y \end{array}\right. \qquad i = 1,2,3,4.$$

sketch of proof: Beginning with the inequalities of (32) one solves for y in terms of x and p. Since we are only concerned with the points (x,y) which lie in $\widetilde{N}$, we ignore those values of y which are less than zero. This leads directly to the function g. □

A similar proof, based on the inequalities (33), leads to the following lemma.

Lemma 7.8. *If* $(x,y) \in \mathrm{Inv}(\widetilde{N})$, *then*

$$g(x, p_{2i-1}, b) \leq y \leq g(x, p_{2i}, b) \qquad i = 1,2,3,4.$$

As always our goal is to show that $\widetilde{N}$ is an isolating neighborhood, i.e. that $\mathrm{Inv}(\widetilde{N}) \cap \partial\widetilde{N} = \emptyset$. In fact, as will be seen for these parameter values $\widetilde{N}$ is an isolating block. Thus, given Lemmas 7.4 and 7.5 it is sufficient to show that if $(x,y) \in \sigma_i$ for some $i = 1, \ldots, 8$, then $H^2(x,y) \notin \widetilde{N}$.

As a first step observe that $\frac{\partial f}{\partial p} > 0$ and $\frac{\partial g}{\partial p} < 0$. Also, for $a = 1.8$ and $0.015 \leq b \leq 0.025$

$$\begin{aligned} \frac{\partial f}{\partial b}(p_i, p_j, b) &= \frac{2(1-p_i) + (bp_j(2+a)}{2\sqrt{\frac{bp_j+1-x}{a}}} \\ &> 0 \end{aligned}$$

while

$$\frac{\partial g}{\partial b}(p_i, p_j, b) = \frac{p_i}{2\sqrt{\frac{1+bp_i-p_j}{a}}}.$$

The important observation is that f and g are monotone in b. Furthermore, for p_i, $i = 1,2,3,4$, they slope in opposite directions, while for p_i, $i = 5,6,7,8$, and $0.015 \leq b \leq 0.025$, $\frac{\partial f}{\partial b}(p_i, p_j, b) > \frac{\partial g}{\partial b}(p_i, p_j, b)$ (this last statement reduces to solving for the sign of a cubic in b over the interval in question). Therefore, to show that (32) and (33) are not simultaneously satisfied for any point $x \in \sigma_i$, $i = 1, \ldots 8$, is reduced to evaluating f and g at the points

$(p_i, p_j, 0.015)$ and $(p_i, p_j, 0.025)$. Doing this results in the following inequalities:

$$
\begin{aligned}
&f(p_1,p_8,b) < g(p_1,p_8,b)\\
&g(p_2,p_5,b) < f(p_2,p_1,b) < f(p_2,p_8,b) < g(p_2,p_4,b)\\
&g(p_3,p_5,b) < f(p_3,p_1,b) < f(p_3,p_8,b) < g(p_3,p_4,b)\\
&g(p_4,p_1,b) < f(p_4,p_1,b)\\
&g(p_5,p_1,b) < f(p_5,p_1,b)\\
&g(p_6,p_5,b) < f(p_6,p_1,b) < f(p_6,p_8,b) < g(p_6,p_4,b)\\
&g(p_7,p_5,b) < f(p_7,p_1,b) < f(p_7,p_8,b) < g(p_7,p_4,b)\\
&f(p_8,p_8,b) < g(p_8,p_8,b),
\end{aligned}
$$

and hence, $\widetilde{N}$ is an isolating block.

7.D. Concluding Remarks.

The proof of Theorem 7.1 was essentially done by continuation. As was discussed in the introduction, this is typical of many applications of fixed point theory and more generally of most types of topological existence results. The continuation was done along the line of parameter values $a = 1.8$ and $0 \leq b \leq 0.025$. However, the standard local continuation theory for the Conley index extends the results to an open neighborhood of this line.

The same analysis can obviously be done for different values of $a = 1.8$. However, as one decreases a one needs to take higher powers of H in order for the corresponding one dimensional map F to exhibit the horseshoe dynamics as explicitly as it occurs at $a = 1.8$. Similarly, the analysis can be extended to larger values of b. However, $\widetilde{N}$ ceases to be an isolating block, and hence, to show that $\sigma_i \cap \mathrm{Inv}(\widetilde{N}) = \emptyset$ one needs to take higher iterates of h_k and h_k^{-1}. For higher iterates, the inequality formulas which correspond to f and g become analytically intractable (the problem reduces to solving high order polynomials). Numerically, it is easy to check that $\widetilde{N}$ remains an isolating neighborhood for values of b well in excess of 0.1. Furthermore, using interval arithmetic one could undoubtably prove such a result.

We have chosen not to pursue these directions of extending Theorem 7.1 at this moment, in part because we feel that the most important challenge is finding theoretical results with which to attack the entire Henon attractor.

References

[Co] C. C. Conley, *Isolated invariant sets and the Morse index*, CBMS no.38, A.M.S., Providence, R.I., 1978.

[De] R. Devaney, *An Introduction to Chaotic Dynamics*, Benjamin–Cummings 1986.

[Do] A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, Berlin Heidelberg New York 1972.

[E1] R. W. Easton, Isolating blocks and symbolic dynamics, *Journal of Differential Equations* 17(1975),96-118.

[E2] R. W. Easton, Isolating blocks and epsilon chains for maps, *Physica D*, 39(1989),95-110.

[GH] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields*, Springer Verlag, New York, 1983.

[Ma] W. S. Massey, *Homology and Cohomology Theory*, Marcel Dekker, Inc., New York and Basel, 1978.

[MMC] C. McCord and K. Mischaikow, On the global dynamics of attractors for scalar delay equations, preprint CDSNS92-89.

[Mi] K. Mischaikow, Global asymptotic dynamics of gradient-like bistable equations, *SIAM Math. Anal.* (to appear).

[MiMo] K. Mischaikow and Y. Morita, Dynamics on the Global Attractor of a Gradient Flow Arising from the Ginzburg-Landau Equation, *JJIAM*, **11** (1994) 185–202.

[Mo] J. Moser, *Stable and Random Motion in Dynamical Systems*, Princton Univ. Press, Princeton, 1973.

[Mr1] M. Mrozek, Index pairs and the fixed point index for semidynamical systems with discrete time, *Fundamenta Mathematicae* 133(1989) 179-194.

[Mr2] M. Mrozek, Leray Functor and the Cohomological Conley Index for Discrete Dynamical Systems,*Transactions of the American Mathematical Society* 318(1990) 149-178.

[Mr3] M. Mrozek, The Morse equation in Conley' index theory for homeomorphisms, *Topology and its Applications* 38(1991),45-60.

[Mr4] M. Mrozek, Shape index and other indices of Conley type for local maps on locally compact Hausdorff spaces, preprint CDSNS92-106.

[RS] J.W. Robbin and D. Salamon, Dynamical systems, shape theory and the Conley index, *Ergodic Theory and Dynamical Systems* 8*(1988), 375-393.

[RSZ] J.W. Robbin, D.A. Salamon and E.C. Zeeman, Dynamical Systems, shape theory and the Conley index. Part III: Morse inequalities and zeta functions, in preparation.