CONSTRUCTION AND PROPERTIES OF THE CONLEY INDEX

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1. Introduction

Let (X, d) be a given locally compact metric space and let

$$\pi : X \times T \to X$$

be a dynamical system on X with continuous time ($T = \mathbb{R}$) or discrete time ($T = \mathbb{Z}$). For $t \in T$ let

$$\pi_t : X \ni x \to \pi(x, t) \in X$$

denote the t-translation map. Let $f := \pi_1$. Note that $\pi(x, n) = f^n(x)$ for $n \in \mathbb{Z}$. Thus in the discrete case the map π may be reconstructed from f. Therefore in the sequel we will identify a discrete dynamical system π with its 1-translation map f. For $A \subset \mathbb{Z}$ we will use the notation $f^A(x) := \{f^n(x) \mid n \in A\}$.

Let $N \subset X$ be a compact subset. The *maximal invariant subset* of N is defined by

$$\operatorname{Inv}(N, \pi) := \operatorname{Inv} N := \{x \in N \mid \forall t \in T \ \pi_t(x) \in N\}.$$

We say that $S \subset X$ is *invariant* if $\operatorname{Inv} S = S$. A compact set $N \subset X$ is called an *isolating neighbourhood* if $\operatorname{Inv} N \subset \operatorname{int} N$. The set S which admits an isolating neighbourhood N such that $S = \operatorname{Inv} N$ is called an *isolated invariant set*. Proving the existence of compact invariant sets as well as understanding their structure and behaviour under perturbation are important goals of the theory of dynamical systems. An important special case is an isolated invariant set. Charles Conley (see [1]) showed that the properties of an isolated invariant set may be studied by investigating the isolating neighbourhood. The advantage is the fact that isolating neighbourhoods are easier to locate and understand. Conley showed that every isolating neighbourhood N carries some information which depends only on the isolated invariant set S in N. The information may be used to derive certain properties of S and may be computed from the topology of certain subsets of N. Depending on

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the particular construction the information may take the form of a certain group, algebra, homotopy type of a space or just an object in an abstract category. Regardless of its particular form the information is referred to as the Conley index.

In this paper we will present a concise but relatively general construction of the Conley index. In the sequel we restrict our attention to discrete dynamical systems. This is not an essential restriction: as was shown in [3] the flow case may be reduced to the discrete case by studying iterates of the t-translation map of the flow for any $t \neq 0$.

An isolating neighbourhood N always admits an *index pair*, i.e. a pair $P = (P_1, P_2)$ of compact subsets of N such that the following three conditions are satisfied

(i) positive invariance relative to N:
$x \in P_i,\ f(x) \in N \ \Rightarrow\ f(x) \in P_i, \ \ i = 1, 2$
(ii) exit set: $x \in P_1,\ f(x) \notin N \ \Rightarrow\ x \in P_2$
(iii) isolation: $\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2)$.

In fact one can prove a slightly more general result:

Theorem 1.1. *Let N be an isolating neighbourhood for f and W a neighbourhood of* $\operatorname{Inv} N$. *Then there exists an index pair P for N with $P_1 \backslash P_2 \subset W$.*

We will present a proof of this theorem in Section 3.

As we already mentioned Conley observed that although there are many different index pairs for a given isolating neighbourhood, they do contain some common information. In the case of a flow the common information is the homotopy type of the quotient space P_1/P_2, where the quotient space P_1/P_2 is obtained as follows. First we take a universal point $* \notin P_1$ and introduce an equivalence $\equiv_P$ in $P_1 \cup \{*\}$ given by

$$x \equiv_P y \text{ iff } x = y \text{ or } x, y \in P_2 \cup \{*\}.$$

Then we take the equivalence classes of $\equiv_P$ as elements in P_1/P_2 and define the open sets in P_1/P_2 as sets whose counter image in the quotient map is open in $P_1 \cup \{*\}$. In the sequel the equivalence class of $x \in P_1 \cup \{*\}$ in the relation $\equiv_P$ will be denoted by $[x]_P$ and the space P_1/P_2 will be treated as a pointed space with the base point being $[*]_P$.

In the discrete case the extraction of the common information is slightly more complicated. First we need to observe that for every index pair P there is an induced map on the quotient space P_1/P_2 defined by

$$f_P([x]_P) := \begin{cases} [f(x)]_P & \text{if } x \in f^{-1}(P_1) \\ [*]_P & \text{otherwise.} \end{cases}$$

It is an easy exercise to verify that f_P is continuous and $f_P([*]_P) = [*]_P$. It turns out that if f comes from a flow then f_P is homotopic to the identity on P_1/P_2 (see [3]). This is why we can disregard f_P in the case of a flow. In the discrete case the common information may be extracted from $(P_1/P_2, f_P)$. This requires introducing the category of endomorphisms.

Let $\mathcal{E}$ be a category. We define the category of endomorphisms of $\mathcal{E}$, denoted by $\mathrm{Endo}(\mathcal{E})$ as follows: the objects of $\mathrm{Endo}(\mathcal{E})$ are pairs (A, a), where $A \in \mathcal{E}$ and $a \in \mathcal{E}(A, A)$ is an endomorphism of A. The set of morphisms from $(A, a) \in \mathcal{E}$ to $(B, b) \in \mathcal{E}$ is the subset of $\mathcal{E}(A, B)$ consisting of exactly those morphisms $\varphi \in \mathcal{E}(A, B)$ for which $b\varphi = \varphi a$. We write $\varphi : (A, a) \to (B, b)$ to denote that φ is a morphism from (A, a) to (B, b) in $\mathrm{Endo}(\mathcal{E})$. Note that in particular $a : (A, a) \to (A, a)$ is a morphism in $\mathrm{Endo}(\mathcal{E})$.

Let $\mathcal{C}$ be another category and $L : \mathrm{Endo}(\mathcal{E}) \to \mathcal{C}$ be a functor. We say that L is *normal* if $L(a)$ is an isomorphism in $\mathcal{C}$ for any endomorphism $a : A \to A$ in $\mathcal{E}$. The reason we are interested in normal functors is the following theorem.

Theorem 1.2. *Assume $L : \mathrm{Endo}(\mathcal{E}) \to \mathcal{C}$ is a normal functor and $\varphi : (A, a) \to (B, b)$, $\psi : (B, b) \to (A, a)$ are such that $a = \varphi\psi$, $b = \psi\varphi$. Then we have the commutative diagram*

$$\begin{array}{ccc} L(A,a) & \xrightarrow{L(a)} & L(A,a) \\ {\scriptstyle L(\varphi)}\big\downarrow & \nearrow L(\psi) & \big\downarrow{\scriptstyle L(\varphi)} \\ L(B,b) & \xrightarrow[L(b)]{} & L(B,b) \end{array}$$

in $\mathcal{C}$, in which all morphisms are isomorphisms.

Proof: The commutativity of the diagram is obvious. The equality $L(\psi)L(\varphi) = L(a)$ implies

$$\Big(L(a)^{-1}L(\psi)\Big)L(\varphi) = \mathrm{id}_{L(A,a)}$$

and $L(\varphi)L(\psi) = L(b)$ implies

$$L(\varphi)\Big(L(\psi)L(b)^{-1}\Big) = \mathrm{id}_{L(B,b)}\,.$$

Hence $L(\varphi)$ is an isomorphism. The proof for $L(\psi)$ is similar. □

It turns out that normal functors are useful in extracting from an isolating neighborhood information dependent only on the isolated invariant set inside. The first step however is taking homotopy. Let Comp_* denote the category of pointed compact metric spaces and let $\mathrm{HtpComp}_*$ denote the homotopy category of pointed compact metric spaces. We recall that objects in both categories are pointed compact spaces (compact spaces with a selected base point) and morphisms in Comp_* are continuous maps preserving the base point whereas morphisms in $\mathrm{HtpComp}_*$ are homotopy classes of such maps. There is also the homotopy functor $\mathrm{Htp} : \mathrm{Comp}_* \to \mathrm{HtpComp}_*$ which fixes objects and sends every base point preserving continuous map $\varphi : X \to Y$ to its homotopy class denoted by $[\varphi]$.

Let $L : \mathrm{Endo}(\mathrm{HtpComp}_*) \to \mathcal{C}$ be a normal functor. (We will show in the next section that every category admits a normal functor.)

Theorem 1.3. *Let S be an isolated invariant set, N be an isolating neighbourhood for S and P be an index pair in N. Then $C(S, f) := C(N, f) := L(P_1/P_2, [f_P])$ is independent of the choice of N and an index pair P in N.*

The object $C(S, f)$ given by the above theorem is called the *Conley index of S*.

2. The Szymczak functor

In order to construct a category $\mathcal{C}$ and a normal functor $L : \mathrm{Endo}(\mathcal{E}) \to \mathcal{C}$ we want to modify the category $\mathrm{Endo}(\mathcal{E})$ in such a way that every morphism $e : (E, e) \to (E, e)$ becomes an isomorphism. Thus let's speculate and see what happens if we add an artificial inverse of e denoted by $\bar{e}$. If $\varphi : (E, e) \to (F, f)$ is a morphism in $\mathrm{Endo}(\mathcal{E})$, then the equality $\varphi \circ e = f \circ \varphi$ implies that also

$$\bar{f} \circ \varphi = \varphi \circ \bar{e}. \tag{1}$$

To preserve the category structure for every sequence $\{\varphi_i\}_{i=1,n}$ of morphisms $\varphi_i : (E_{i-1}, e_{i-1}) \to (E_i, e_i)$ and any sequence $\{k_i\}_{i=0,n}$ of natural numbers we need to add the composition

$$\bar{e}_n^{k_n} \circ \varphi_n \circ \bar{e}_{n-1}^{k_{n-1}} \circ \cdots \circ \varphi_1 \circ \bar{e}_0^{k_0}.$$

However it follows from (1) that the above composition equals

$$\varphi_n \circ \varphi_{n-1} \circ \cdots \varphi_1 \circ \bar{e}_0^{n_k + n_{k-1} + \ldots + n_0}.$$

Thus what we actually need to add are all morphisms of the form

$$\varphi \circ \bar{e}^m \tag{2}$$

for any morphism $\varphi : (E, e) \to (F, f)$ and any natural number m. Notice that the morphism $\bar{e}$ is uniquely determined by the morphism φ. Therefore it is natural to identify the morphism (2) with the pair (φ, m). The formula (2) implies that the definition of the composition of the added morphisms should read

$$(\psi, k) \circ (\varphi, l) := (\psi \circ \varphi, k + l),$$

where $\varphi : (E,e) \to (F,f)$, $\psi : (F,f) \to (G,g)$ and $k,l \in \mathbb{N}$. However there is one problem with the above definition: it may happen that for $\varphi, \varphi' : (E,e) \to (F,f)$ we have:

$$\varphi \circ e^{m'+k} = \varphi' \circ e^{m+k} \tag{3}$$

for some $m, m', k \in \mathbb{N}$. Then multiplying formula (3) on the right hand side by $\bar{e}^{m+m'+k}$ we obtain

$$\varphi \circ \bar{e}^{m} = \varphi' \circ \bar{e}^{m'}, \tag{4}$$

i.e.

$$(\varphi, m) = (\varphi', m').$$

Thus we need to introduce an equivalence relation. Namely we write that

$$(\varphi, m) \equiv (\varphi, m') \tag{5}$$

whenever there exists a $k \in \mathbb{N}$ such that the condition (3) is satisfied. One easily verifies that this indeed defines an equivalence relation in $\operatorname{Endo}(\mathcal{E})\big((E,e),(F,f)\big)$. The Szymczak category of endomorphisms of $\mathcal{E}$, $\operatorname{Szym}(\mathcal{E})$ can be now defined formally as follows. The objects of $\operatorname{Szym}(\mathcal{E})$ are the same as in $\operatorname{Endo}(\mathcal{E})$. The morphisms in $\operatorname{Szym}(\mathcal{E})\big((E,e),(F,f)\big)$ are the equivalence classes of the relation (5). One easily checks that the definition of composition carries over to the equivalence classes. The identity morphism on $(E,e) \in \operatorname{Szym}(\mathcal{E})$ is $[\mathrm{id}_E, 0]$. There is a functor $\operatorname{Szym} : \operatorname{Endo}(\mathcal{E}) \to \operatorname{Szym}(\mathcal{E})$ which fixes objects and sends a morphism $\varphi : (E,e) \to (F,f)$ to the equivalence class $[\varphi, 0]$. In general it may happen that $\operatorname{Szym}(\varphi) = \operatorname{Szym}(\varphi')$ even if $\varphi \neq \varphi'$. Nevertheless it is convenient to write just φ to denote $\operatorname{Szym}(\varphi)$ whenever it is clear from the context in which category we work. Now we can formally define the inverse of $e : (E,e) \to (E,e)$ in $\operatorname{Szym}(\mathcal{E})$ by

$$\bar{e} := [\mathrm{id}_E, 1].$$

Indeed, we have

$$e \circ \bar{e} = [e, 0] \circ [\mathrm{id}_E, 1] = [e, 1] = [\mathrm{id}_E, 0] = \mathrm{id}_{(E,e)}$$

which shows that $\bar{e}$ is an inverse of e. We can also write the abstract morphism $[\varphi, n]$ in terms of $\bar{e}$ as

$$[\varphi, n] = [\varphi, 0] \circ [\mathrm{id}_E, 1]^n = \varphi \circ \bar{e}^n.$$

Obviously $\operatorname{Szym}(e)$ is invertible in $\operatorname{Szym}(\mathcal{E})$, therefore Szym is a normal functor. Actually this is the most general normal functor. The reader may easily show that any other normal functor $L : \operatorname{Endo}(\mathcal{E}) \to \mathcal{C}$ admits a unique functor $L' : \operatorname{Szym}(\mathcal{E}) \to \mathcal{C}$ such that $L = L' \circ \operatorname{Szym}$.

The construction of the Szymczak category and the Szymczak functor is due to Szymczak [8].

3. Construction of index pairs

Construction of index pairs is probably the most technical part of the Conley index theory. There are several different proofs available (see [1],[5],[6],[7]). We will present here a proof which goes back to some early constructions. First we recall the following straightforward proposition.

Proposition 3.1. *Assume $N \subset X$ is compact. Then the set* $\operatorname{Inv} N$ *is compact and so are the sets*

$$\operatorname{Inv}^- N := \{x \in N \mid \forall n \in \mathbb{Z}^-\ f^n(x) \in N\},$$

$$\operatorname{Inv}^+ N := \{x \in N \mid \forall n \in \mathbb{Z}^+\ f^n(x) \in N\}.$$

To guide the reader's intuition, we sketch the proof first, developing the necessary lemmas. The idea is to take as the first approximation for P_1 a compact neighbourhood A of $\operatorname{Inv}^- N$ and as the first approximation of P_2 the set $P_1 \setminus U$, where U is an open neighbourhood of $\operatorname{Inv}^+ N$. Such a pair does meet the isolation condition but obviously it need not meet the relative positive invariance condition. Therefore we enlarge the sets by following them along trajectories as long as they stay in N. Formally this is done as follows. For $x \in N$ we put

$$f_N^+(x) := \{y \in N \mid \exists j \in \mathbb{Z}\ y = f^j(x) \text{ and } f^{[0,j]}(x) \subset N\}.$$

This actually defines a multivalued map. We recall that a multivalued map is a map $F : X \to \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ stands for the family of all subsets of Y. If $F : X \to \mathcal{P}(Y)$ is a multivalued map and $A \subset X$ then the image of A under F is defined by $F(A) := \bigcup_{x \in A} F(x)$. Thus $f_N^+ : N \ni x \to f_N^+(x) \in \mathcal{P}(N)$ is a multivalued map. We put

$$P_1 := f_N^+(A),$$

$$P_2 := f_N^+(P_1 \setminus U).$$

The first question now is if the modified sets are still compact. They would be compact if f_N^+ were an upper semicontinuous map. A multivalued map $F : X \to \mathcal{P}(Y)$ is upper semicontinuous if for every $x \in X$ and every open set U such that $F(x) \subset U$ there exists an open neighbourhood V of x such that $y \in V$ implies $F(y) \subset U$. A well known fact is that the image of a compact set under a compact valued upper semicontinuous map is compact.

Unfortunately f_N^+ need not be upper semicontinuous. However, for every fixed $n \in \mathbb{N}$ the map $f_{N,n} : N \to \mathcal{P}(N)$ given by

$$f_{N,n}(x) := \{y \in N \mid \exists j \in [0, n]\ y = f^j(x) \text{ and } f^{[0,j]}(x) \subset N\}$$

is upper semicontinuous:

Proposition 3.2. *The map $f_{N,n} : N \to \mathcal{P}(N)$ is upper semicontinuous for any $n \in \mathbb{N}$.*

Proof: Suppose that $f_{N,n}$ is not upper semicontinuous. Then there exists a point $x \in N$ and an open subset U of N satisfying $f_{N,n}(x) \subset U$ and a convergent sequence $x_k \to x$, $\{x_k\} \subset N$ with $f_{N,n}(x_k) \not\subset U$. Consequently for each k, there exists an $m_k \in [0, n]$ such that $f^{m_k}(x_k) \in N \setminus U$. Passing to a subsequence we may assume that $m_k = m \in [0, n]$ for all k. It follows that $f^m(x) \notin U$, a contradiction. □

One can easily see that

$$f_N^+(x) := \bigcup_{n \in \mathbb{Z}^+} f_{N,n}(x),$$

unfortunately, upper semicontinuity does not carry over to infinite unions. However, if $x \notin \operatorname{Inv}^+ N$, then the above union is obviously finite and one can expect that this property carries over to compact sets. Indeed, we have the following lemma.

Lemma 3.3. *Let $K \subset N$ be a compact subset of X such that $K \cap \operatorname{Inv}^+ N = \emptyset$. Then*

(a) *$f_{N,n}(K) = \emptyset$ for all but finitely many $n \in \mathbb{N}$,*
(b) *The mapping f_N^+ restricted to K is upper semicontinuous,*
(c) *$f_N^+(K) \cap \operatorname{Inv}^+ N = \emptyset$.*

Proof: Since $x \in K$ implies $x \notin \operatorname{Inv}^+ N$, for each $x \in K$ there exists an $n_x \in \mathbb{Z}^+$ such that $f_{N,n_x}(x) = \emptyset$. Since f_{N,n_x} is upper semicontinuous, there exists a neighbourhood V_x of x such that $f_{N,n_x}(V_x) = \emptyset$. Let $\{V_{x_1}, \dots V_{x_k}\}$ be a finite covering of K. Put

$$m := \max\{n_{x_i} \mid i = 1, 2, \dots k\}.$$

Then $f_{N,n}(K) = \emptyset$ for all $n \geq m$. This proves (a). Property (b) follows from (a) and Proposition 3.2 since the union of finitely many upper semicontinuous maps is upper semicontinuous. Property (c) is straightforward. □

Returning to our construction we see that $P_2 = f_N^+(P_1 \setminus U)$ is compact, because $K := P_1 \setminus U$ satisfies the assumptions of the above lemma. What concerns P_1, we have the following lemma.

Lemma 3.4. *If $\operatorname{Inv}^- N \subset A \subset N$ and A is compact then $f_N^+(A)$ is compact.*

Proof: It is sufficient to show that $f_N^+(A)$ is closed. Let $\{y_k\}$ be a sequence of points in $f_N^+(A)$, $y_k \to y \in N$. We need to show that $y \in f_N^+(A)$. Let $x_k \in A$, $n_k \in \mathbb{Z}^+$ be such that $y_k = f^{n_k}(x_k)$ and $f^{[0,n_k]}(x_k) \subset N$. We will consider first the case when $\{n_k\}$ is bounded. Then, passing to a subsequence if necessary, we may assume that $n_k = n$ for all k. Since N and A are compact, we conclude that $y = f^n(x)$ and $f^{[0,n]}(x) \subset N$, i.e. $y \in f_N^+(A)$. Hence it remains to consider the case when $\{n_k\}$ is unbounded. Then, passing to a subsequence, we may assume that $\{n_k\}$ is increasing and, by restricting the interval $[0, n_k]$, that $n_k = k$. It follows that for any $m \in \mathbb{Z}^-$ and almost all $k \in \mathbb{Z}^+$ $f^m(y_k) \in N$, hence also $f^m(y) \in N$. Thus $y \in \operatorname{Inv}^- N \subset A$. □

Thus we have a pair (P_1, P_2) of compact set, which are positively invariant with respect to N. The exit set property is equivalent to showing that $P_1 \setminus P_2 \subset f^{-1}(N)$. Since $\operatorname{Inv} N$ is invariant under f, one can expect that points close to $\operatorname{Inv} N$ cannot be mapped by f outside of N. Thus the exit set property will be guaranteed if we choose A and U in such a way that $P_1 \setminus P_2$ is close to $\operatorname{Inv} N$. Actually, this stronger postulate is a part of the assertion of Theorem 1.1 anyway. To fulfill it, given a neighbourhood W of $\operatorname{Inv} N$, we take U an open neighbourhood of $\operatorname{Inv}^+ N$ and V an open neighbourhood of $\operatorname{Inv}^- N$ such that $U \cap V \subset W$. This is always possible, because $\operatorname{Inv} N = \operatorname{Inv}^+ N \cap \operatorname{Inv}^- N$. Then $P_1 \setminus U \subset f_N^+(P_1 \setminus U) = P_2$, which implies that $P_1 \setminus P_2 \subset U$. To guarantee that $P_1 \setminus P_2 \subset W$ it is sufficient now to show that one can choose A in such a way that $P_1 = f_N^+(A) \subset V$. This is the contents of the following lemma.

Lemma 3.5. *For any neighbourhood V of $\operatorname{Inv}^- N$ there exists a compact neighbourhood A of $\operatorname{Inv}^- N$ such that $f_N^+(A) \subset V$.*

Proof: Similarly to the map $f_{N,n}$ for any $n \in \mathbb{N}$ we define the map $f_{N,-n} : N \to \mathcal{P}(N)$ given by

$$f_{N,-n}(x) := \{y \in N \mid \exists j \in [-n, 0]\ y = f^j(x) \text{ and } f^{[j,0]}(x) \subset N\}$$

The set $K := N \setminus V$ is disjoint from $\operatorname{Inv}^- N$. An argument dual to the proof of Lemma 3.3 shows that there exists an $m \in \mathbb{Z}^+$ satisfying

$$f_{N,-m}(N \setminus V) = \emptyset. \tag{6}$$

Basically the same proof as in Proposition 3.2 shows that $f_{N,-m}$ is upper semicontinuous. Therefore one can find for every $x \in \operatorname{Inv}^- N$ a compact neighbourhood V_x of x such that $f_{N,-m}(V_x) \subset V$. Let $\{V_{x_1}, \dots V_{x_k}\}$ be a finite covering of $\operatorname{Inv}^- N$. Put $A := \bigcup_{i=1}^{k} V_{x_i}$. Then A is a compact neighbourhood of $\operatorname{Inv}^- N$ such that $f_{N,-m}(A) \subset V$. It remains to show that $f_N^+(A) \subset V$. Indeed, let $y \in f_N^+(A)$. Then there exists an $n > 0$

and an $x \in A$ such that $y \in f_{N,n}(x)$. If $n \leq m$, we are done. If $n > m$, we note that $x \in f_{N,-n}(y)$, hence $f_{N,-m}(y) \neq \emptyset$ and (6) implies that $y \in V$. □

We are now ready to summarize the proof of the existence of index pairs.

Proof of Theorem 1.1 Since $\operatorname{Inv} N = f(\operatorname{Inv} N)$ is a compact subset of $\operatorname{int} N$, we may assume that $W \cup f(W) \subset \operatorname{int} N$. Let A be given for V by Lemma 3.5. We define

$$P_1 := f_N^+(A),\ P_2 := f_N^+(P_1 \backslash U). \tag{7}$$

Then $P_1 \subset V$ and $P_1 \backslash U \subset P_2$ which implies that $P_1 \backslash P_2 \subset U$. Therefore $P_1 \backslash P_2 \subset U \cap V \subset W$. We verify that (P_1, P_2) is an index pair. P_1 is compact by Lemma 3.4 and P_2 is compact by Lemma 3.3 (b), since $P_1 \backslash U$ is compact and the image of a compact set in an upper semicontinuous compact valued map is compact. Obviously, $P_2 \subset f_N^+(P_1) \subset P_1$. Positive invariance of P_1 and P_2 relative to N is obvious. Since $P_1 \backslash P_2 \subset W$ and $f(W) \subset \operatorname{int} N$, we see that the exit set property is satisfied. In order to verify the isolation property, observe that P_1 is a neighbourhood of $\operatorname{Inv}^- N$ and, by Lemma 3.3 (c), $N \backslash P_2$ is a neighbourhood of $\operatorname{Inv}^+ N$. Therefore $P_1 \backslash P_2 = P_1 \cap (N \backslash P_2)$ is a neighbourhood of $\operatorname{Inv}^- N \cap \operatorname{Inv}^+ N = \operatorname{Inv} N$. □

4. Correctness of the definition

There are several different proofs of the correctness of the construction of the Conley index. What one has to prove is that given an isolated invariant set S, its two isolating neighbourhoods N and M and index pairs $P = (P_1, P_2)$, $Q = (Q_1, Q_2)$ respectively in N and M, the objects $L([P_1/P_2], [f_P])$ and $L([Q_1, Q_2], [f_Q])$ are isomorphic. One way to prove this is to construct explicitly the isomorphism (see [8]). However the explicit formula for the isomorphism is complicated and it is of no use once the proof is completed. Therefore we will choose here another approach, which follows the ideas in [4]. Again, to guide the reader's intuition, we sketch the proof first, developing the necessary lemmas.

The first observation is that it is relatively easy to write down the isomorphism if $M = N$ and the index pairs P and Q do not lie too far apart. To be more specific, assume that $P \subset Q$ and

$$f(Q_i) \cap N \subset P_i \text{ for } i = 1, 2. \tag{8}$$

Then one can define the map $f_{QP} : Q_1/Q_2 \to P_1/P_2$ by

$$f_{QP}([x]_Q) := \begin{cases} [f(x)]_P & \text{if } x \in f^{-1}(N) \\ [*] & \text{if } x \in Q_2 \cup \{*\}. \end{cases}$$

and the map $\iota_{PQ} : P_1/P_2 \to Q_1/Q_2$ by

$$\iota_{PQ}([x]_P) := [x]_Q.$$

The reader will check easily that both maps are well defined and continuous. We have the following lemma.

Lemma 4.1. *Assume that $P \subset Q$ are index pairs in N such that (8) is satisfied. If L : $\mathrm{Endo}(\mathrm{HtpComp}_*) \to \mathcal{C}$ is a normal functor then $L([\iota_{PQ}])$ is an isomorphism in $\mathcal{C}$.*

Proof: One easily verifies that the following diagram

$$\begin{tikzcd} P_1/P_2 \arrow[r, "f_P"] \arrow[d, "\iota_{PQ}"] & P_1/P_2 \arrow[d, "\iota_{PQ}"] \\ Q_1/Q_2 \arrow[r, "f_Q"] \arrow[ur, "f_{QP}"] & Q_1/Q_2 \end{tikzcd}$$

commutes. Applying the homotopy functor and Theorem 1.2 we get the required conclusion. □

The next step is to get rid of the assumption (8). This may be achieved by finding a sequence of index pairs $P = Q^n \subset Q^{n-1} \subset \cdots \subset Q^1 \subset Q^0 = Q$ such that any two consecutive index pairs in the sequence satisfy condition (8). The sequence may be constructed recursively by pushing forward the larger index pair into the smaller index pair with the map f. Unfortunately the procedure requires an extra assumption to guarantee that the construction really produces a new index pair, for instance it suffices to assume that the index pairs P, Q differ by at most one coordinate. We will see later on that there is no problem in getting rid of that assumption in the next step. The following two lemmas formalize the described procedure.

Lemma 4.2. *Let $P \subset Q$ be index pairs for N which differ by at most one coordinate, i.e. $P_1 = Q_1$ or $P_2 = Q_2$. Define a pair of sets $G(P, Q)$ by*

$$G_i(P, Q) = P_i \cup (f(Q_i) \cap N), \text{ for } i = 1, 2. \tag{9}$$

Then

(a) *If $P_i = Q_i$ then $G_i(P, Q) = P_i = Q_i$, for $i = 1, 2$.*
(b) *$P \subset G(P, Q) \subset Q$;*
(c) *$G(P, Q)$ is an index pair*
(d) *$f(Q_i) \cap N \subset G_i(P, Q)$, for $i = 1, 2$.*

Proof: Property (d) is obvious and property (a) is immediate from the positive invariance of P_i, Q_i relative to N. The first inclusion in (b) is obvious and the second is an immediate consequence of the positive invariance of Q_i relative to N. It remains to prove (c). To prove the positive invariance relative to N let $x \in G_i(P,Q)$ and $y := f(x) \in N$. If $x \in P_i$ then obviously $y \in G_i(P,Q)$. If $x \in f(Q_i) \cap N$ then $x \in Q_i$ hence $y \in f(Q_i) \cap N \subset G_i(P,Q)$. To prove the exit set property let us note that

$$G_1(P,Q)\backslash G_2(P,Q) \subset Q_1 \backslash G_2(P,Q) \subset Q_1 \backslash P_2$$

and since P and Q differ by at most one coordinate either $Q_1\backslash P_2 = Q_1\backslash Q_2 \subset f^{-1}(N)$, or $Q_1\backslash P_2 = P_1\backslash P_2 \subset f^{-1}(N)$. Finally observe that $\operatorname{Inv} N \subset \operatorname{int}(P_1\backslash P_2) \cap \operatorname{int}(Q_1\backslash Q_2)$ so the isolation property will follow if we verify that $(P_1\backslash P_2) \cap (Q_1\backslash Q_2) \subset G_1(P,Q)\backslash G_2(P,Q)$. Indeed, let $y \in (P_1\backslash P_2) \cap (Q_1\backslash Q_2)$. Then $y \in G_1(P,Q)$ and it remains to show that $y \notin f(Q_2) \cap N$. However if $y \in f(Q_2) \cap N$, then $y \in Q_2$, a contradiction. □

Lemma 4.3. *Let $P \subseteq Q$ be index pairs which differ by at most one coordinate. Then there exists a sequence of pairs*

$$P = Q^n \subset Q^{n-1} \subset \cdots \subset Q^1 \subset Q^0 = Q$$

with the following properties

(a) *If $P_i = Q_i$ then $Q_i^k = P_i = Q_i$ for all $k = 1, 2, \ldots n-1; i = 1, 2$.*
(b) *Q^k is an index pair for all $k = 1, 2, \ldots n-1$.*
(c) *$f(Q_i^k) \cap N \subset Q_i^{k+1}, i = 1, 2; k = 0, 1, \ldots n-1$.*

Proof: Let Q^k be given by the recurrence formula $Q^0 := Q$, $Q^{k+1} := G(P, Q^k)$, $k \in \mathbb{N}$. By Lemma 4.2 and induction on k,$\{Q^k\}$ is a decreasing sequence of index pairs containing P and satisfying (a), (b) and (c) for all $k \in \mathbb{Z}$. It remains to show that $Q^n = P$ for some n. Indeed, suppose that the inclusion $P \subset Q^k$ is strict for all $k \in \mathbb{N}$, i.e. if $i \in \{1,2\}$ is such that $P_i \neq Q_i$ then $Q_i^k \backslash P_i \neq \emptyset$. Fix a $k \in \mathbb{N}$ and choose an $x_k \in Q_i^k \backslash P_i$. We will show by induction in j that

$$f^{-j}(x_k) \in Q_i^{k-j} \setminus P_i \text{ for } j = 0, 1, \ldots, k. \tag{10}$$

The assertion is obvious for $j = 0$. If $f^{-j}(x_k) \in Q_i^{k-j} \setminus P_i$, then $f^{-j}(x_k) \in f(Q_i^{k-j-1}) \cap N$, i.e. $f^{-(j+1)}(x_k) \in Q_i^{k-(j+1)}$. It cannot be $f^{-(j+1)}(x_k) \in P_i$, because otherwise $f^{-j}(x_k) \in P_i$. Thus the assertion (10) is proved. Put $u_k := f^{-k}(x_{2k})$. It follows from (10) that

$$f^{[-k,k]}(u_k) \subset Q_i \setminus P_i \subset Q_i \setminus \operatorname{int} P_i. \tag{11}$$

Taking a subsequence we may assume that $u_n \to u \in Q_i \setminus \operatorname{int} P_i$ and it follows from (11) that $f^j(u) \in Q_i \setminus \operatorname{int} P_i$ for $j \in \mathbb{Z}$, i.e. $u \in \operatorname{Inv}(Q_i \setminus \operatorname{int} P_i)$.

If $i = 1$ we get $u \in \operatorname{Inv}(Q_1 \setminus \operatorname{int} P_1) \subset \operatorname{Inv} N \subset \operatorname{int}(P_1 \setminus P_2) \subset \operatorname{int} P_1$, a contradiction. If $i = 2$, we get $u \in \operatorname{Inv}(Q_2 \setminus \operatorname{int} P_2) \subset \operatorname{Inv} Q_2$. On the other hand $\operatorname{Inv} Q_2 \subset \operatorname{Inv} N \subset \operatorname{int}(Q_1 \setminus Q_2) \subset Q_1 \setminus Q_2$, which implies that $\operatorname{Inv} Q_2 = \emptyset$, a contradiction again. □

The following proposition is a straightforward observation which will let us get rid of the assumption that P and Q differ by at most one coordinate.

Proposition 4.4. *If $P \subseteq Q$ are index pairs for N, then so are $(P_1, P_1 \cap Q_2)$ and $(P_1 \cup Q_2, Q_2)$.*

In order to consider index pairs which not necessarily are included one in the other, we need the following lemma.

Lemma 4.5. *If P and Q are index pairs for N, then so is $P \cap Q$.*

Proof: Positive invariance relative to N and exit set property are obvious. To prove the isolation property let us note that $\operatorname{int}(P_1 \setminus P_2) \cap \operatorname{int}(Q_1 \setminus Q_2) \subset \operatorname{int}(P_1 \cap Q_1 \setminus (P_2 \cup Q_2)) \subset \operatorname{int}(P_1 \cap Q_1 \setminus P_2 \cap Q_2)$. □

Finally, we will also need the following straightforward proposition.

Proposition 4.6. *Assume $P \subset Q$ are compact pairs such that $P_1 \setminus P_2 = Q_1 \setminus Q_2$. Then P_1/P_2 and Q_1/Q_2 are homeomorphic.* □

We are ready now to present the proof of Theorem 1.3.

Proof of Theorem 1.3 We need to show that if M and N are two isolating neighbourhoods of S, P an index pair for N and Q an index pair for M then $L(P_1/P_2, [f_P])$ and $L(Q_1/Q_2, [f_Q])$ are isomorphic in $\mathcal{C}$. The proof will be given in several steps.

Step 1. We begin with the following special case

(i) $M = N$,
(ii) $P \subset Q$,
(iii) P and Q differ by at most one coordinate
(iv) $f(Q_i) \cap N \subset P_i$, for $i = 1, 2$.

The conclusion in this case follows immediately from Lemma 4.1.

Step 2. We lift the assumption (iv). By Lemma 4.3 we can take a sequence $\left\{Q^k\right\}_{k=0,1\ldots n}$ of index pairs such that $Q_0 = Q$, $Q_n = P$ and the pair of index pairs $Q^{k+1} \subseteq Q^k$ satisfies the assumptions (i) through (iv). The conclusion follows.

Step 3. We lift the assumption (iii). Put $R_1 := P_1 \cup Q_2, R_2 := P_1 \cap Q_2$. By Proposition 4.4 (P_1, R_2) and (R_1, Q_2) are index pairs. We have the commutative diagram of inclusions

$$\begin{array}{ccc} (P_1, R_2) & \xrightarrow{j_2} & (R_1, Q_2) \\ \uparrow i_1 & & \downarrow i_3 \\ (P_1, P_2) & \xrightarrow{j} & (Q_1, Q_2) \end{array}$$

It is clear that the pair of index pairs $(P_1, P_2) \subset (P_1, R_2)$ satisfies the assumption (iii) and so does $(R_1, Q_2) \subset (Q_1, Q_2)$, so by Step 2, the inclusions i_1 and i_3 induce isomorphisms in $\mathcal{C}$. Since $P_1 \backslash R_2 = P_1 \backslash Q_2 = R_1 \backslash Q_2$, it follows from Proposition 4.6 that the inclusion j_2 induces an isomorphism in $\mathcal{C}$.

Step 4 Now, only (i) is assumed. By Lemma 4.5 $P \cap Q$ is an index pair, hence the conclusion of Step 3 can be applied to pairs $P \cap Q \subset P$ and $P \cap Q \subset Q$.

Step 5 If $M \neq N$, one may always assume that $M \subset N$ since otherwise $M \cap N$ can be considered which is also an isolating neighbourhood of S. By Step 4, it is sufficient to show the existence of one index pair P for N and one index pair Q for M such that $L(P_1/P_2, [f_P])$ and $L(Q_1/Q_2, [f_Q])$ are isomorphic.

By Theorem 1.1 there exists an index pair P for N such that $P_1 \backslash P_2 \subset \operatorname{int} M \cap f^{-1}(\operatorname{int} M)$. It is easily verified that $Q := (M \cap P_1, M \cap P_2)$ is an index pair for M. Since $Q_1 \backslash Q_2 = M \cap (P_1 \backslash P_2) = P_1 \backslash P_2$, the inclusion $Q \subset P$ induces an isomorphism in $\mathcal{C}$ by Proposition 4.6. □

5. Properties of the Conley index.

The Conley index has several properties which facilitate its computation.

Theorem 5.1. Ważewski property: *If N is an isolating neighbourhood with respect to f and $\operatorname{Con}(N, f) \neq \operatorname{Con}(\emptyset, f)$ then $\operatorname{Inv} N$ is nonempty.*

Proof: If $\operatorname{Inv} N = \emptyset$, then $(\emptyset, \emptyset)$ is an index pair for N. It follows that $\operatorname{Con}(N, f) = \operatorname{Con}(\emptyset, f)$. □

The following two properties are studied in detail in [2].

Theorem 5.2. Stability property: *If N is an isolating neighbourhood with respect to f then it is an isolating neighbourhood with respect to g for g sufficiently close to f in the compact-open topology and the Conley indexes with respect to f and g coincide.*

Theorem 5.3. Homotopy property: *Assume f_s, $s \in [0, 1]$, is a family of maps $f_s : X \to X$, continuously depending on s. If $N \subset X$ is an isolating neighbourhood with respect to every map f_s then $\operatorname{Con}(N, f_s)$ does not depend on s.*

Theorem 5.4. Additivity property: *Assume the category $\mathcal{C}$ admits the product, denoted $\times$ and L maps products into products. If N, N_1, N_2 are isolating neighbourhoods such that* $\operatorname{Inv} N = \operatorname{Inv} N_1 \cup \operatorname{Inv} N_2$, $\operatorname{Inv} N_1 \cap \operatorname{Inv} N_2 = \emptyset$ *then*

$$\operatorname{Con}(N, f) = \operatorname{Con}(N_1, f) \times \operatorname{Con}(N_2, f)$$

Proof: Put $S_i := \operatorname{Inv} N_i$ and select disjoint open neighbourhoods U_1, U_2 of S_1 and S_2 respectively. For $i = 1, 2$ let M_i be a compact neighbourhood of S_i such that $N_i \subset U$. Select P^i, index pairs for S_i in M_i. One can easily verify that $P := P^1 \cup P^2$ is an index pair for $\operatorname{Inv} N$ in $M := M_1 \cup M_2$. Obviously $f_P = f_{P^1} \cup f_{P^2}$. Hence, $(P_1/P_2, [f_P])$ is the wedge sum of $(P_1^1/P_2^1, [f_{P^1}])$ and $(P_1^2/P_2^2, [f_{P^2}])$. It suffices now to apply L. $\square$

Theorem 5.5. Normalization property: *The whole space X is an isolating neighbourhood and* $\operatorname{Con}(X, f) = L([X], [f])$.

Proof: The conclusion follows from the fact that $(X, \emptyset)$ is an index pair in X. $\square$

Theorem 5.6. Commutativity property *(see* [4]*). Assume $f = \psi\phi, g = \phi\psi$, where $\phi : X \to Y, \psi : Y \to X$ are continuous,. If $S \subset X$ is an isolated invariant set with respect to f then $\phi(S)$ is an isolated invariant set with respect to g and* $\operatorname{Con}(S, f) = \operatorname{Con}(\phi(S), g)$.

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