Conley Index Theory

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1 Introduction

Given the goals and constraints of this handbook a complete self contained introduction to the Conley index theory is not possible. Therefore, these notes will focus on that part of the theory which is most closely related to the study of the dynamics of differential equations. We will work in the context of flows on locally compact metric spaces. In this setting most of the essential ideas can be explained without too much technical difficulty. Of course the disadvantage to this approach is that the results are not directly applicable to the dynamics generated by partial and functional differential equations. There are two responses to this. First, having read this survey the reader is encouraged to turn to Rybakowski's book [74] where the theory is generalized to semi-flows on infinite dimensional spaces. Second, many physically interesting infinite dimensional systems are dissipative and contain global compact attractors. Furthermore, upon restricting the dynamics to these attractors one typically has a flow. This, of course, is precisely the setting of the index theory described herein.

With this in mind we adopt the convention that throughout this paper X will denote a locally compact metric space with metric μ. Furthermore, $\varphi : \mathbf{R} \times X \to X$ will represent a *flow*; that is a continuous map satisfying:

$$\varphi(0, x) = x$$

$$\varphi(t,\varphi(s,x)) \quad = \quad \varphi(t+s,x)$$

A set $S \subset X$ is an *invariant set* for the flow φ if

$$\varphi(\mathbf{R},S) := \bigcup_{t\in\mathbf{R}} \varphi(t,S) = S.$$

Much of the qualitative theory of differential equations involves the study of the existence and structure of invariant sets. There are at least four difficulties associated with the investigation of nonlinear systems:

1. Invariant sets are global objects, and therefore, their study involves obtaining global estimates of a nonlinear problem.

2. Invariant sets posses an infinite variety of different structures some of which can be extremely complicated (chaotic dynamics and fractal structures).

3. The structure of invariant sets can change dramatically with respect to perturbations (bifurcation theory, normal forms, catastrophe theory);

4. The bifurcation points need not be isolated (structurally stable systems are not dense).

The fact that for a fixed family of differential equations all of these problems must be considered simultaneously makes the analysis of nonlinear systems difficult.

Consider for instance the simplest example of an invariant set, that of a *fixed point* or *equilibrium*, i.e. a point $x \in X$ such that $\varphi(\mathbf{R},x) = x$. For many differential equations proving the existence of fixed points either analytically or numerically is a nontrivial task. Degree theory is by now a standard tool for this type of problem. The degree of a map can be thought of as a function from a set of continuous maps to the integers with three important properties:

1. It is defined in terms of a region for which there are no fixed points on the boundary.

2. If the degree of a map is nonzero, then the map possesses a fixed point.

3. Degree is a continuous function.

To successfully apply this theory requires a few essential steps. Direct computation of the degree of a map is often quite difficult. Therefore, the first step is to choose a continuous family of maps going from the map of interest to a simpler map for which the fixed points are explicitly known. One then chooses a region containing one or more equilibria of the simple map for which the resulting degree is nonzero. The third step typically involves analysis; one must show that each function in the continuous family of maps does not posses a fixed point on the boundary of the region. Having done this one observes that since the degree is integer valued and continuous, it cannot change throughout the continuous family of maps. Therefore, the degree of the original map of interest is the same as that of the simpler map. In particular the degree is nonzero, and hence, the original map has a fixed point. The power of this approach arises from the fact that one does not need to study the actual behavior of the map on the interior of the region.

This short digression into degree theory was included since the application of the Conley index typically involves a similar process. The regions of interest are called isolating neighborhoods. A compact set $N \subset X$ is an *isolating neighborhood* if

$$\mathrm{Inv}(N, \varphi) := \{x \in N \mid \varphi(\mathbf{R}, x) \subset N\} \subset \mathrm{int}\, N.$$

where int N denotes the interior of N. S is an *isolated invariant set* if $S = \mathrm{Inv}(N)$ for some isolating neighborhood N.

The most important property of an isolating neighborhood is that it is robust with respect to perturbation. To state this more precisely consider a continuous family of dynamical systems

$$\varphi_\lambda : \mathbf{R} \times X \to X, \quad \lambda \in [-1, 1]. \tag{1}$$

Proposition 1.1 *Let N be an isolating neighborhood for the flow φ_0. Then, for sufficiently small $\delta > 0$, N is an isolating neighborhood for all φ_λ, $|\lambda| < \delta$.*

The proof of this proposition is fairly straight forward; the essential ideas being as follows. Since N is an isolating neighborhood, $x \in \partial N$ implies that $x \notin \mathrm{Inv}(N, \varphi_0)$. Equivalently, there exists a time $t_x > 0$ such that $\varphi([-t_x, t_x], x) \not\subset N$. The compactness of N implies that $T := \max_{x \in \partial N} t_x < \infty$. By continuity of the flows, one may approximate φ_0 over the time interval $[-T, T]$ arbitrarily closely by choosing δ sufficiently small. This robustness of isolating neighborhoods with respect to perturbations leads to the following definition.

Definition 1.2 Let $N \subset X$ be a compact set. Let $S_\lambda = \mathrm{Inv}(N, \varphi_\lambda)$. Two isolated invariant sets S_{λ_0} and S_{λ_1} are *related by continuation* or S_{λ_0} *continues to* S_{λ_1} if N is an isolating neighborhood for all φ_λ, $\lambda \in [\lambda_0, \lambda_1]$.

Observe that in itself the definition of continuation is about isolating neighborhoods. It says nothing about the associated isolated invariant set. For example if S_{λ_0} is a degenerate fixed point, then it typically is related by continuation to the isolated invariant set consisting of the empty set. It is in this sense that the inherent difficulties of bifurcation theory are avoided.

Of course what we are interested in is the structure and properties of invariant sets not isolating neighborhoods. Therefore we need a means by which we can pass from knowledge of isolating neighborhoods to an understanding of its associated invariant set. This is the purpose of the Conley index. The index will be defined in Section 3. However, for the moment, consider its important characteristics.

1. The Conley index is an index of isolating neighborhoods. Furthermore, if N and N' are isolating neighborhoods for the flow φ and $\operatorname{Inv}(N,\varphi) = \operatorname{Inv}(N',\varphi)$, then the Conley index of N is the same as the Conley index of N'. Observe, that this implies that we can also consider the Conley index as an index of isolated invariant sets. We shall make use of both interpretations depending on which is most convenient.

2. *(Ważewski Property)* If the Conley index of N is not trivial, then $\operatorname{Inv} N \neq \emptyset$.

3. *(Continuation)* If N is an isolating neighborhood for a continuous parameterized family of flows φ_λ, $\lambda \in [0,1]$, that is

$$\operatorname{Inv}(N,\varphi_\lambda) \subset \operatorname{int} N \quad \text{for } \lambda \in [0,1],$$

then the Conley index of N under φ_0 is the same as the Conley index of N under φ_1.

The applicability of the Conley index depends essentially on these three properties. The first gives great freedom in the choice of regions in phase space on which one will perform the analysis. The second allows one to pass from the isolating neighborhood to an understanding of the dynamics of the isolated invariant set. The Ważewski property, while the most fundamental, is the simplest result of this type. Section 4 contains a variety of more sophisticated theorems which can been used to prove the existence of connecting orbits, periodic orbits, and even chaotic dynamics in the sense of symbolic dynamics. The third property is important for the following reason. The Conley index is a purely topological index and as such it is a very coarse measure of the dynamics. Typically, if it can be computed directly at a particular parameter value, then one's knowledge of the dynamics at that parameter value is reasonably complete. The power of the index (as in degree theory) comes from being able to continue it to a parameter value where ones understanding of the dynamics is much less complete.

So far the discussion of the Conley index has been restricted to dynamical systems generated by flows. However, the index theory can also be extended to the setting of maps. Even in the context of differential equations this extension is important for at least two reasons. First, there are many situations in which the flow admits a Poincaré section for which one can study the dynamics of the associated Poincaré map. As will be made clear in Section 4.2, the Conley index for the Poincaré map carries essentially more information than the index for the flow. The second reason arises from the fact that numerical approximations to the flow can take the form of maps.

Again, to avoid as many technical details as possible we will consider discrete dynamical systems that are generated by homeomorphisms $f : X \to X$. If one begins with a flow $\varphi : \mathbf{R} \times X \to X$, then natural way to generate a discrete dynamical system is to take the time τ map of the flow $f : X \to X$ defined by $f(x) = \varphi(\tau, x)$.

Several ideas from the Conley index for flows carry over directly to the discrete case. The most fundamental is that of an isolating neighborhood. Given $N \subset X$ the maximal invariant set of N is defined by

$$\operatorname{Inv}(N, f) := \{ x \in N \mid f^n(x) \in N \text{ for all } n \in \mathbf{Z} \}.$$

A compact set N is called an *isolating neighborhood* if $\operatorname{Inv}(N, f) \subset \operatorname{int} N$ and a set S is an *isolated invariant set* if there exists an isolating neighborhood N such that $S = \operatorname{Inv}(N, f)$.

Another set of ideas that are identical in the continuous and discrete case involves the notion of decompositions of isolated invariant sets. These issues are discussed in Section 2 where the results are all stated in the context of flows. The reader, however, should feel confident in replacing flow by homeomorphism throughout that section.

The real difference between the setting of flows and homeomorphisms comes in the definition of the index. This will be explained in greater detail in Section 3.2. However, the essential difference is that if $f : X \to X$ is the time τ map of a flow as defined above, then f is homotopic to the identity map on X. This is not the case for an arbitrary homeomorphism. Never the less, the Conley index for maps as described in these notes is a true generalization of the Conley index for flows. If S is an isolated invariant set for a flow φ, then S will be an isolated invariant set for the associated time τ map, and furthermore, the Conley indices of these two objects will agree.

In Section 5, yet another, framework for the index theory will be developed; that of multivalued maps. While the theory is relatively new, its inspiration is not. One of the fundamental theorems in dynamical systems is the decomposition theorem of Conley which states that any compact invariant set can be divided into its chain recurrent part and the rest. Furthermore, on the latter part one can define a strictly decreasing Lyapunov function. These two portions are therefore, precisely the sets on which one has recurrent dynamics and gradient-like dynamics. This result appears as Theorem 2.19 of these notes. However, in the definition of the chain recurrent set one moves forward by the flow for a minimal time and then is allowed to make an error of size $\epsilon > 0$. This is the simplest form of a multivalued map; points get sent to sets, in this case the ϵ neighborhood around the true image. A slightly more complicated method for generating multivalued maps is to use a numerical scheme to approximate the trajectories of a differential equation and to keep track of error estimates or, better yet, error bounds. Observe that this produces a map that provides an a priori outer estimate for the dynamics of the differential equation. Furthermore, if the numerical estimates are good, then the set of possible single valued maps that are approximated by the multivalued map will be close to one another in the C^0 norm. However, this is precisely the setting in which the Conley index continues. Therefore, one might expect that the Conley theory for multivalued maps might provide a means by which numerical computations could be used to obtain index information for differential equations. This is in fact the case as will be made clear in Section 5. The final section contains a sample of the types of numerical results that have been obtain .

We conclude this introduction by returning to the comments in the opening paragraph. The goal of these notes is to indicate how the Conley index theory can be used to understand the structure of dynamics arising from applications. This precludes a discussion of the important influence that Conley's ideas had in geometry and topology, in particular in the development of Floer homology and the proof of the Arnold conjecture. For this the reader is referred to Conley and Zehnder's original work [15, 16], the series of papers by Floer [19, 20, 21, 22, 23] or some recent books on the subject [38, 66].

2 Decompositions of Invariant Sets

As was indicated earlier the purpose of the Conley index theory is to understand the structure of isolated invariant sets. To describe this structure requires the ability to decompose invariant sets into subinvariant sets. Since one of the properties of the index theory is its stability under perturbation, it is important that the method of decomposition also be stable with respect to perturbation.

2.1 Attractor–Repeller Pairs

The coarsest decomposition of an invariant set is that of an attractor–repeller pair. Our discussion begins with some standard definitions from the theory of dynamical systems. Let $Y \subset X$. The *omega limit set* of Y is

$$\omega(Y) = \omega(Y, \varphi) := \bigcap_{t>0} \mathrm{cl}\left(\varphi([t,\infty), Y)\right)$$

while the *alpha limit set* of Y is

$$\alpha(Y) = \alpha(Y, \varphi) := \bigcap_{t>0} \mathrm{cl}\left(\varphi((-\infty,-t], Y)\right).$$

Definition 2.1 Let S be a compact invariant set. $A \subset S$ is an *attractor* in S if there exists a neighborhood U of A such that

$$\omega(U \cap S) = A.$$

The *dual repeller* of A in S is

$$R := \{x \in S \mid \omega(x) \cap A = \emptyset\}.$$

The pair (A, R) is called an *attractor–repeller pair decomposition* of S.

The following properties of attractors and repellers are easy to check.

Proposition 2.2 *Let S be an isolated invariant set. Let (A, R) be an attractor–repeller pair decomposition for S. Then*

1. *A and R are isolated invariant sets;*

2. if A' is an attractor in A, then A' is an attractor in S.

Example 2.3 Consider the ordinary differential equation

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= cy + (\lambda - x^2), \quad c \in \mathbf{R},\ \lambda \in [-1,1].\end{aligned}$$

Let $N = [-\kappa, \kappa] \times [-\kappa, \kappa] \subset \mathbf{R}^2$. It is fairly easy to show that for sufficiently large κ, N is an isolating neighborhood for all $c \in \mathbf{R}$ and $\lambda \in [-1,1]$. Let $S_{c,\lambda} = \mathrm{Inv}(N)$ for the given parameter values. Define $H(x,y) := y^2/2 + x^3/3 - \lambda x$, then $dH/dt = cy^2$. So for $c \neq 0$, H acts as a Lyapunov function.

For the moment let us restrict our attention to $c > 0$ and $0 < \lambda \leq 1$. Then there are two equilibria in N, $(\pm\sqrt{\lambda}, 0)$. Since $H(\sqrt{\lambda}, 0) > H(-\sqrt{\lambda}, 0)$ we can conclude that $((\sqrt{\lambda}, 0), (-\sqrt{\lambda}, 0))$ is an attractor repeller pair decomposition of $S_{c,\lambda}$.

From the definition of an attractor–repeller pair decomposition it is clear that if $x \in S$ then it is impossible for $\omega(x) \subset R$ and $\alpha(x) \subset A$, i.e. there can be no connecting orbits from the attractor to the repeller. Of course, connections in the other direction may exist. The set of *connecting orbits* from R to A in S is denoted by

$$C(R, A; S) := \{x \in S \mid \omega(x) \subset A,\ \alpha(x) \subset R\}.$$

The following result, though trivial, is important enough to be designated a theorem.

Theorem 2.4 *Let (A, R) be an attractor–repeller pair decomposition of S. Then*

$$S = A \cup R \cup C(R, A; S).$$

In Example 2.3 an attractor–repeller pair decomposition was constructed. As will be demonstrated in Section 4.1, in appropriate situations the Conley index can be used to show that $C(R, A; S) \neq \emptyset$, i.e that connecting orbits exist.

Theorem 2.5 *Attractor-repeller pair decompositions continue.*

Proof. Assume that S_0 is an isolated invariant set with isolating neighborhood N. Then, by Proposition 1.1 there exists $\lambda_S > 0$ such that N is an isolating neighborhood for all φ_λ, $\lambda \in [-\lambda_S, \lambda_S]$. Let (A_0, R_0) denote an attractor repeller pair decomposition of S_0. Since A_0 and R_0 are isolated invariant sets, they have isolating neighborhoods $N_A \subset N$ and $N_R \subset N$. Furthermore, there exists $\lambda_A > 0$ and $\lambda_R > 0$ such that N_A is an isolating neighborhood for $\lambda \in [-\lambda_A, \lambda_A]$ and N_R is an isolating neighborhood for $\lambda \in [-\lambda_R, \lambda_R]$.

Let $\lambda_0 = \min\{\lambda_S, \lambda_A, \lambda_R\}$. Thus, for all $\lambda \in [-\lambda_0, \lambda_0]$, the $A_\lambda := \mathrm{Inv}(N_A, \varphi_\lambda)$ are related by continuation, the $R_\lambda := \mathrm{Inv}(N_R, \varphi_\lambda)$ are related by continuation, and the S_λ are related by continuation.

The final point that needs to be checked is that there exists $0 < \lambda_1 \leq \lambda_0$ such that for all $\lambda \in [-\lambda_1, \lambda_1]$, (A_λ, R_λ) is an attractor repeller pair for S_λ. Assume not. Then there exists a sequence of parameter values $\lambda_n \to 0$ and points $x_n \in S_{\lambda_n}$ such that $x_n \notin A_{\lambda_n} \cup R_{\lambda_n}$ and $\omega(x_n) \notin A_{\lambda_n}$. Invoking the continuity of the family of flows and the fact that (A_0, R_0) is an attractor repeller pair decomposition of S_0 gives a contradiction. Thus, there exists an interval $[-\lambda_1, \lambda_1]$ over which the attractor repeller pair decomposition continues. ∎

Example 2.6 Let us return now to Example 2.3 and examine the question of continuation. To begin with, since N is an isolating neighborhood for all values of $c \in \mathbf{R}$ and $\lambda \in [-1, 1]$ the corresponding isolated invariant sets $S_{c,\lambda}$ are related by continuation. Furthermore, if we restrict as above to $c > 0$ and $0 < \lambda \leq 1$, then the attractor repeller pair decomposition $((\sqrt{\lambda}, 0), (-\sqrt{\lambda}, 0))$ also continues. However, it does not continue over the full parameter space of $c \in \mathbf{R}$ and $\lambda \in [-1, 1]$. In particular, the attractor repeller pair decomposition breaks down in two ways in this example. The first is that at $\lambda = 0$ the two equilibria merge and for $\lambda < 0$, $S_{c,\lambda} = \emptyset$. The second way occurs for $\lambda > 0$ when $c = 0$. In this case the system is Hamiltonian and the integral curves consisting of elements of $S_{c,\lambda}$ extend from $(\sqrt{\lambda}, 0)$ to $(-\sqrt{\lambda}, 0)$.

In the above examples a Lyapunov function was used to obtain the existence of an attractor-repeller pair decomposition. In fact, as the following theorem indicates they are in some sense equivalent concepts.

Theorem 2.7 *Let S be a compact invariant set with attractor repeller pair (A, R). Then, there exists a continuous function*

$$V : S \to [0, 1]$$

such that

(i) $V^{-1}(1) = R$

(ii) $V^{-1}(0) = A$

(iii) for $x \in C(R, A)$ and $t > 0$, $V(x) > V(\varphi(t, x))$.

Proof. Observe that if A or R equals the empty set, then the result is trivially true. So assume that $A \neq \emptyset \neq R$. The proof is now broken down into three steps. The first is to define a function $f : S \to [0, 1]$ by

$$f(x) = \frac{\mu(x, A)}{\mu(x, A) + \mu(x, R)}$$

Clearly, (since $A \cap R = \emptyset$) f is continuous, $f^{-1}(0) = A$, and $f^{-1}(1) = R$.

Second, define $g : S \to [0, 1]$ by

$$g(x) = \sup_{t \geq 0} \{f(\varphi(t, x))\}.$$

Again, it is obvious that $g^{-1}(0) = A$, and $g^{-1}(1) = R$ and $g(\varphi(t,x)) \leq g(x)$ for all $t \geq 0$. It is left to the reader to show that g is continuous.

The third and final step is to define

$$V(x) = \int_0^\infty e^{-\xi} g(\varphi(\xi,x))d\xi$$

Clearly (i) and (ii) are satisfied. If $x \in C(R,A)$, then

$$\begin{aligned} V(x) - V(x,t) &= \int_0^\infty e^{-\xi} g(\varphi(\xi,x))d\xi - \int_0^\infty e^{-\xi} g(\varphi(\xi+t,x))d\xi \\ &= \int_0^\infty e^{-\xi}[g(\varphi(\xi,x)) - g(\varphi(\xi+t,x))]d\xi \\ &\geq 0. \end{aligned}$$

Now $V(x) - V(\varphi(t,x)) = 0$ if and only if $g(\varphi(\xi,x)) - g(\varphi(\xi+t,x)) = 0$ for all $\xi \geq 0$, i.e., $g(\varphi(\xi,x)) = \bar{g}$ a constant for all $\xi \geq 0$. This implies that $\omega(x) \cap (A \cup R) = \emptyset$, a contradiction. Thus $V(x) - V(\varphi(t,x)) > 0$, and condition (iii) is satisfied. ∎

2.2 Morse Decompositions

Example 2.8 Consider the following modification of Example 2.3.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= cy + (x^2-1)(x+1/2), \quad c > 0. \end{aligned} \tag{2}$$

As before, $N = [-\kappa,\kappa] \times [-\kappa,\kappa] \subset \mathbf{R}^2$ is an isolating neighborhood for large κ. This system still possesses a Lyapunov function $H(x,y) := y^2/2 - (x^4/4 + x^3/6 - x^2/2 - x/2)$. However, in this case there are three equilibria $(-1,0), (-1/2,0), (1,0)$ and it would be nice to decompose the isolated invariant set S_c into three isolated subsets. This requires a generalization of the concept of an attractor repeller pair.

In an attractor repeller pair (A,R) decomposition of S there is a natural total ordering that can be imposed on these two isolated invariant subsets given by $R > A$, implying that there is no connecting orbit from A to R. If one has a multitude of isolated invariant subsets, then it is more natural to use partial orders. A *partial order*[1] on a set $\mathcal{P}$ is a relation $>$ which satisfies:

(i) $p > p$ never holds for $p \in \mathcal{P}$.

(ii) If $p > q$ and $q > r$ then $p > r$.

If, in addition, the partial order satisfies

[1] Actually we are defining a *strict partial order*, however, to save the use of an unnecessary adjective we shall use the above definition.

(iii) for all $p, q \in \mathcal{P}$, either $p > q$ or $q > p$.

Then $>$ is called a *total order*. ¿From now on $(\mathcal{P}, >)$ will be used to denote a *finite* indexing set $\mathcal{P}$ with a partial order $>$.

Definition 2.9 A finite collection

$$\mathcal{M}(S) = \{M(p) \mid p \in \mathcal{P}\}$$

of disjoint compact invariant subsets of S is a *Morse decomposition* if there exists a partial order $>$ on the indexing set $\mathcal{P}$ such that for every

$$x \in S \setminus \bigcup_{p \in \mathcal{P}} M(p)$$

there exists $p, q \in \mathcal{P}$ such that $p > q$ and

$$\omega(x) \subset M(q) \quad \text{and} \quad \alpha(x) \subset M(p).$$

The sets $M(p)$ are called *Morse sets*. Observe that it is not assumed that there is unique order on $\mathcal{P}$. In fact, any ordering on $\mathcal{P}$ with the above property is called *admissible*. Having chosen an admissible order $>$ we shall often write

$$\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}.$$

Strictly speaking this is an abuse of notation since the Morse decomposition only refers to the collection of Morse sets. In applications, however, the construction of a Morse decomposition typically involves knowledge of a Lyapunov function, which can be used to define an admissible order.

Example 2.10 Returning to Example 2.8, let $M(3) := (-1/2, 0)$, $M(2) = (-1, 0)$ and $M(1) := (1, 0)$. Then, for $c > 0$ we have the Morse decomposition

$$\mathcal{M}(S_c) = \{M(i) \mid i = 1, 2, 3,\ 3 > 2 > 1\}$$

where the ordering is determined by the Lyapunov function H. Observe that $>$ is an admissible ordering for all $c > 0$. This is not, however, the only partial ordering that is admissible. Consider equation (2) at the two extreme parameter values $c = 0$ and $c = \infty$.

At $c = 0$ we have a Hamiltonian system where the set of bounded solutions has two components: one consisting of the point $(1, 0)$; and the other containing $(-1, 0)$, $(-1/2, 0)$, a variety of periodic orbits, and an orbit homoclinic to $(-1, 0)$. From this one can conclude that for $0 < c << 1$ there can be no connecting orbit between $(1, 0)$ and the other two equilibria. Therefore, an admissible order is $3 >_0 2$ with 1 unrelated to either 2 or 3.

A simple rescaling of time allows one to write down $\dot{x} = -(x^2 - 1)(x + 1/2)$ as the limiting equation as $c \to \infty$. For this equation it is easy to check that the set of connecting orbits consist of heteroclinic orbits from $(-1/2, 0)$ to $(\pm 1, 0)$. Thus, for c sufficiently large one can use the ordering $3 >_\infty 2$ and $3 >_\infty 1$.

In both these examples the order obtained by an explicit study of the flow at a given parameter value is different from that obtained using the Lyapunov function. In particular, $>$ contains more relations that either $>_0$ and $>_\infty$, and therefore, less information about the structure of the dynamics.

This example leads to the observation that while there may be many admissible flows for a given Morse decomposition, there is, a unique minimal (in the sense of the number of order relations) admissible order which is called the *flow defined order*. Recall that an order $>'$ on $\mathcal{P}$ is an *extension* of $>$ if $p > q$ implies $p >' q$. Thus, any admissible order must be an extension of the flow defined order. In applications one typically does not know the flow defined order, and therefore, must work with an extension.

As in the case of attractor repeller pair decompositions, Morse decompositions are strongly related to the existence of Lyapunov functions. The following result is a direct generalization of Theorem 2.7.

Theorem 2.11 *Let S be an isolated invariant set. Let*

$$\mathcal{M}(S) = \{M(p) \mid p \in \mathcal{P}\}$$

be a finite collection of disjoint invariant compact subsets of S. Then, $\mathcal{M}(S)$ is a Morse decomposition of S if and only if there exists a continuous function $V : S \to [0,1]$ such that:

$$\begin{array}{rcl} \forall\, x, y \in M(p) & \textit{implies} & V(x) = V(y) \\ x \in S \setminus \bigcup_{p\in\mathcal{P}} M(p) & \textit{implies} & V(x) > V(\varphi(t,x)) \quad \forall t > 0 \end{array}$$

Observe that given a Morse decomposition $\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}$ one can always coarsen it. To do this systematically we introduce the following notation. A subset $I \subset \mathcal{P}$ is called an *interval* if $p, q \in I$ and $p > r > q$ implies that $r \in I$. The set of intervals on $(\mathcal{P}, >)$ will be denoted by $\mathcal{I}(\mathcal{P}, >)$.

Proposition 2.12 *Let $I \in \mathcal{I}(\mathcal{P}, >)$ and define*

$$M(I) = \left(\bigcup_{p\in I} M(p)\right) \cup \left(\bigcup_{p,q\in I} C(M(p), M(q))\right).$$

Then, $M(I)$ is an isolated invariant set.

The proof is fairly straight forward and follows from the compactness of S.

Proposition 2.13 $\mathcal{M}(S) = \{M(p) \mid p \in \mathcal{P}\backslash I\} \cup \{M(I)\}$ *defines a Morse decomposition of S. Furthermore, an admissible partial order $>'$ is given by*

$$\begin{array}{rcl} p >' q & \Leftrightarrow & p > q \text{ if } p, q \in \mathcal{P}\backslash I, \\ p >' I & \text{if} & \exists\, q \in I \text{ such that } p > q, \\ I >' p & \text{if} & \exists\, q \in I \text{ such that } q > p. \end{array}$$

This section began with the statement that Morse decompositions are generalizations of attractor–repeller pair decompositions. To see this observe that an attractor–repeller pair decomposition (A, R) of S defines a Morse decomposition

$$\mathcal{M}(S) = \{M(p) \mid p = 1, 2, \quad 2 > 1\}$$

where $M(1) = A$ and $M(2) = R$.

In the converse direction, within a Morse decomposition there are many attractor repeller pairs. To make this precise define an *adjacent pair of intervals* in $(\mathcal{P}, >)$ to be an ordered pair (I, J) of mutually disjoint intervals satisfying:

(i) $I \cup J \in \mathcal{I}(\mathcal{P}, >)$,

(ii) $p \in I$, $q \in J$ implies that $p \not> q$.

The collection of adjacent pairs is denoted by $\mathcal{I}_2(\mathcal{P}, >)$. If (I, J) is an adjacent pair, then set $IJ := I \cup J$.

Lemma 2.14 *If $(I, J) \in I_2(\mathcal{P}, >)$, then $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$.*

Example 2.15 Returning to Example 2.10, consider $c \approx \infty$. The flow defined order is $3 >_\infty 2$ and $3 >_\infty 1$. Thus $(2, 3), (3, 1) \in \mathcal{I}_2(\mathcal{P}, >_\infty)$. Therefore, $(M(2), M(3))$ and $(M(1), M(3))$ form attractor–repeller pair decompositions of $M(23)$ and $M(13)$, respectively.

Knowing that Morse decompositions can be coarsened it is natural to ask whether given an isolated invariant set S, there exists a finest Morse decomposition. The following example shows that the answer is no.

Example 2.16 Consider the equation

$$\dot{x} = x^2 \sin \frac{\pi}{x}.$$

Let $N = [-2, 2]$. Then, $\operatorname{Inv} N = S = [-1, 1]$ is a compact isolated invariant set. We can define Morse decompositions as follows. Let

$$\mathcal{M}_n(S) := \{\pm 1/k \mid k = 1, \ldots, n-1\} \cup \{[-1/n, 1/n]\}.$$

For each n this is a Morse decomposition with $2n - 1$ Morse sets and as n increases the Morse decomposition becomes more refined. Since, by definition, a Morse decomposition can only consist of a finite number of Morse sets there is no finest Morse decomposition.

2.3 Conley's Decomposition Theorem

As was indicated by Example 2.16 there need not be a finest Morse decomposition. However, one can still ask the following question. Given a compact invariant set S is there a finest collection of subsets of S off of which one can define a Lyapunov function? One of the fundamental theorems of dynamical systems states that the answer is yes. To present it the notion of chain recurrence is essential.

An (ϵ, τ) *chain* from x to y is a finite sequence

$$\{(x_i, t_i)\} \subset X \times [\tau, \infty), \quad i = 1, \dots n$$

such that $x = x_1$, $t_i \geq \tau$, $\mu(\varphi(t_i, x_i), x_{i+1}) \leq \epsilon$ and $\mu(\varphi(t_n, x_n), y) \leq \epsilon$. If there exists an (ϵ, τ) chain from x to y, then we write $x \succeq_{(\epsilon,\tau)} y$. If $x \succeq_{(\epsilon,\tau)} y$ for all (ϵ, τ), then $x \succeq y$.

Definition 2.17 The *chain recurrent set* of X under the flow φ is defined by

$$\mathcal{R}(X, \varphi) = \{x \in X \mid x \succeq x\}.$$

The chain recurrent set is fundamental for two reasons. First, it is minimal in the sense of the following theorem.

Theorem 2.18 (Conley [7]) *If S is a compact invariant set, then*

$$\mathcal{R}\left(\mathcal{R}(S, \varphi), \varphi|_{\mathcal{R}(S)}\right) = \mathcal{R}(S, \varphi).$$

Second, it captures all the recurrent dynamics as is indicated by the following theorem.

Theorem 2.19 (Conley [7]) *Let $\mathcal{R}_i(S)$, $i = 1, 2, \dots$, denote the connected components of $\mathcal{R}(S)$. Let S be a compact invariant set. Then there exists a continuous function*

$$V : S \to [0, 1]$$

such that:

(i) if $x \notin \mathcal{R}(S)$ and $t > 0$, then $V(x) > V(\varphi(t, x))$;

(ii) for each $i = 1, 2, \dots$, there exists $\sigma_i \in [0, 1]$ such that $\mathcal{R}_i \subset V^{-1}(\sigma_i)$ and, furthermore, the $\{\sigma_i\}$ can be chosen such that $\sigma_i \neq \sigma_j$ if $i \neq j$.

Unfortunately, for direct applications the components of the chain recurrent set are difficult to work with since they are not robust with respect to perturbation. Furthermore, while the individual components are invariant sets they need not be isolated invariant sets.

Example 2.20 Returning to Example 2.16

$$\mathcal{R}(S) = \{\pm\frac{1}{n} \mid n \in \mathbf{Z}\} \cup \{0\},$$

and hence, has an infinite number of components. In addition, observe that 0 is a component of $\mathcal{R}(S)$, but is not an isolated invariant set. Now consider the parameterized system

$$\dot{x} = x^2 \sin\frac{\pi}{x} + \lambda(x^2 - 1) \quad \lambda \in [-1, 1]$$

with corresponding flow φ_λ. Again, choose $N = [-2, 2]$ and let $S_\lambda := \mathrm{Inv}(\varphi_\lambda)$. Then, for $\lambda \neq 0$, $\mathcal{R}(S_\lambda)$ has only finitely many components.

2.4 References

The material of this section is by now classical. The most complete reference is C. Conley's monograph [7], though Salamon's article [75] is also recommended for its clarity of proof. A proof due to C. Robinson of Theorem 2.19 for compact invariant sets in the setting of maps can be found in [72]. See also the exposition in [73]. In the non compact case there is work by M. Hurley [39, 40].

3 Conley Index

In the previous section we discussed the decomposition of isolated invariant sets into Morse decompositions. Furthermore, we provided a series of simple examples in which we were able to determine the existence of an attractor–repeller pair or of a Morse decomposition, but we were unable to say anything about the existence or structure of connecting orbits between these invariant sets. To do this we will need to use the Conley index.

3.1 Flows

Before providing a definition of the index we recall a few elementary notations from topology.

Definition 3.1 A *pointed space* (Y, y_0) is a topological space Y with a distinguished point $y_0 \in Y$. Given a pair (N, L) of spaces with $L \subset N$,

$$N/L := (N \setminus L) \cup [L]$$

where $[L]$ denotes the equivalence class of points in L in the equivalence relation: $x \sim y$ if and only if $x = y$ or $x, y \in L$. In the sequel we will usually use N/L to denote the pointed space $(N/L, [L])$. The topology on $(N/L, [L])$ is defined as follows: a set $U \subset N \setminus L$ is open if U is open in N and $U \cap L = \emptyset$, or the set $(U \cap (N \setminus L)) \cup L$ is open in N. If $L = \emptyset$, then

$$(N/L, [L]) := (N \cup \{*\}, \{*\})$$

where $\{*\}$ denotes the equivalence class consisting of the empty set.

Let (X, x_0) and (Y, y_0) be pointed topological spaces and let

$$f, g : (X, x_0) \to (Y, y_0)$$

be continuous functions. Implicit in this notation is the assumption that $f(x_0) = g(x_0) = y_0$.

Definition 3.2 f is *homotopic* to g, denoted by

$$f \sim g,$$

if there exists a continuous function $F : X \times [0,1] \to Y$ such that

$$\begin{aligned} F(x,0) &= f(x) \\ F(x,1) &= g(x) \\ F(x_0,s) &= y_0, \quad 0 \le s \le 1. \end{aligned}$$

Obviously $\sim$ is an equivalence relation. The equivalence class of f in this relation is called the *homotopy class* of f and denoted $[f]$.

Definition 3.3 Two pointed topological spaces (X, x_0) and (Y, y_0) are *homotopic*

$$(X, x_0) \sim (Y, y_0)$$

if there exists $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (X, x_0)$ such that

$$f \circ g \sim \mathrm{id}_Y \quad \text{and} \quad g \circ f \sim \mathrm{id}_X .$$

Observe that homotopy defines an equivalence class on the set of topological spaces. The reader for whom homotopy is a new concept may wish to check that $\mathbf{R}^2 \setminus \{0\} \sim S^1$.

We now turn to the definition of the Conley index. Recall that φ is a flow on X a locally compact metric space. As was indicated in the introduction, even though the Conley index is an index for isolating neighborhoods we shall define it in terms of the isolated invariant set.

Definition 3.4 Let S be an isolated invariant set. A pair of compact sets (N, L) where $L \subset N$ is called an *index pair* for S if:

1. $S = \mathrm{Inv}(\mathrm{cl}(N \setminus L))$ and $N \setminus L$ is a neighborhood of S.
2. L is *positively invariant* in N; that is given $x \in L$ and $\varphi([0,t],x) \subset N$, then $\varphi([0,t],x) \subset L$.
3. L is an *exit set* for N; that is given $x \in N$ and $t_1 > 0$ such that $\varphi(t_1, x) \notin N$, then there exists $t_0 \in [0, t_1]$ for which $\varphi([0,t_0],x) \subset N$ and $\varphi(t_0, x) \in L$.

The following theorem gives the existence of index pairs.

Theorem 3.5 *Given an isolated invariant set S, there exists an index pair.*

Isolating neighborhoods and index pairs are very general objects; and as such useful for computational purposes since they are relatively easy to find. A much stricter notion is that of an *isolating block*. This is a compact set B such that

$$B^- := \{x \in B \mid \varphi([0,T),x) \not\subset B, \ \forall T > 0\}.$$

is closed and for all $T > 0$

$$\mathrm{Inv}_T(B, \varphi) \subset \mathrm{int}\, B,$$

where

$$\mathrm{Inv}_T(B, \varphi) := \{x \in B \mid \varphi([-T,T],x) \subset B\}.$$

Theorem 3.6 *Given an isolated invariant set S and a smooth flow, there exists an isolating block B such that $S = \operatorname{Inv} B$. Furthermore, (B, B^-) is an index pair for S.*

Definition 3.7 The *homotopy Conley index* of S is

$$h(S) = h(S, \varphi) \sim (N/L, [L]).$$

Observe that the Conley index of S has been defined in terms of any index pair. Furthermore, typically an isolated invariant set possesses a multitude of isolating neighborhoods. Therefore one needs the following theorem.

Theorem 3.8 (The Conley index is well defined) *Let (N, L) and (N', L') be index pairs for an isolated invariant set S. Then*

$$(N/L, [L]) \sim (N'/L', [L']).$$

Observe from the definition that the homotopy Conley index is the homotopy type of a topological space. Unfortunately, working with homotopy classes of spaces is extremely difficult. To get around this, it is useful to consider the homological Conley index defined by

$$CH_*(S) := H_*(N/L, [L]) \approx H_*(N, L).$$

Since the homology of two homotopic spaces is the same, the homology index is well defined.

Remark 3.9 It is not true that given any index pair (N, L), $H_*(N/L, [L]) \approx H_*(N, L)$. However, one can prove that one can always find index pairs for which this isomorphism holds. Since for some of the algebra it is more convenient to work with the pair (N, L) directly, rather than the quotient space we will hence forth assume that we are working with these types of index pairs.

Finally, we state the continuation theorem for the Conley index.

Theorem 3.10 *(*Continuation Property*) Let S_{λ_0} and S_{λ_0} be isolated invariant sets that are related by continuation. Then,*

$$CH_*(S_{\lambda_0}) \approx CH_*(S_{\lambda_1}).$$

Let us now consider some examples of the Conley index.

Example 3.11 Observe that the empty set $\emptyset$ is vacuously an isolated invariant set. Furthermore, $(\emptyset, \emptyset)$ is an index pair for the empty set. Thus

$$CH_*(\emptyset) \approx 0$$

The contrapositive of this example, though trivial, is significant enough implications to be designated as a theorem.

Theorem 3.12 (Ważewski Property) *Let N be an isolating neighborhood and assume that*

$$CH_*(\operatorname{Inv} N) \not\approx 0.$$

Then,

$$\operatorname{Inv} N \neq \emptyset.$$

This result provides the simplest example of an existence result which can be obtained via the Conley index. It also demonstrates an important point concerning the way one wishes to view the Conley index. The computation of the index was done with knowledge of the invariant set. The theorem of interest was stated in terms of the isolating neighborhood.

After the empty set, the simplest isolated invariant sets are hyperbolic fixed points.

Theorem 3.13 *Let S be a hyperbolic fixed point with an unstable manifold of dimension n. Then*

$$CH_k(S) \approx \begin{cases} \mathbf{Z} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $\dot{x} = f(x)$, $x \in \mathbf{R}^{n+m}$ be the ordinary differential equation for which S is a hyperbolic fixed point. By the Hartman-Grobman theorem, the flow in a neighborhood of S is topologically equivalent to the flow in a neighborhood of the origin of

$$\dot{y} = Df(S)y. \tag{3}$$

Thus it suffices to compute the Conley index of the origin under this linear dynamics. A linear change of variables transforms (3) to

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where A is an $m \times m$ matrix for which the real parts of all its eigenvalues are less that zero and B is an $n \times n$ matrix for which the real parts of all its eigenvalues are greater that zero. An isolating neighborhood of the origin is given by $[-1,1]^m \times [-1,1]^n$. The exit set is given by $[-1,1]^m \times \partial([-1,1]^n)$. The result now follows from computing $H_*\left([-1,1]^m \times [-1,1]^n, [-1,1]^m \times \partial([-1,1]^n)\right)$. ∎

Example 3.14 Consider Example 2.3. Linearizing about the equilibria and counting the number of eigenvalues with positive real part one can check that for $c > 0$,

$$CH_k((-\lambda, 0)) \approx \begin{cases} \mathbf{Z} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$CH_k((\lambda, 0)) \approx \begin{cases} \mathbf{Z} & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.15 The same type of argument applied to Example 2.8 gives

$$CH_k((\pm 1, 0)) \approx \begin{cases} \mathbf{Z} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$CH_k((1/2, 0)) \approx \begin{cases} \mathbf{Z} & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

The Thom isomorphism theorem [4, 49, 77] provides a general means of computing the Conley index for normally hyperbolic invariant sets.

Theorem 3.16 *Assume that a manifold S is a normally hyperbolic invariant set. Let E be the vector bundle over S defined by the local unstable manifold of S. If E is a rank n orientable bundle and one uses homology with* field *coefficients, then*

$$CH_k(S) \approx H_{k+n}(S).$$

As a corollary we obtain the following result.

Corollary 3.17 *Let S be a hyperbolic invariant set that is diffeomorphic to a circle. Assume that S has an oriented unstable manifold of dimension $n+1$. Then*

$$CH_k(S) \approx \begin{cases} \mathbf{Z} & \textit{if } k = n, n+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Example 3.18 Observe that from Corollary 3.17 the dynamics on the isolated invariant set S plays no role with respect to $CH_*(S)$. In particular, it implies that the index of S is the same whether S is a hyperbolic periodic orbit; a circle on which there are two hyperbolic fixed points connected by two heteroclinic orbits; or even a circle consisting entirely of fixed points. One can obtain the same conclusion by a continuation argument. However, the point is that given an isolating neighborhood N with maximal invariant set $S = \operatorname{Inv} N$ and

$$CH_k(S) \approx \begin{cases} \mathbf{Z} & \text{if } k = n, n+1 \\ 0 & \text{otherwise.} \end{cases}$$

one may *not* conclude that S contains a periodic orbit.

This is an important issue. Because the Conley index remains invariant under continuation, in general, the index alone does not provide sufficient information to draw conclusions about the structure of the invariant set. The results described in Section 4 are devoted to this problem. What will be shown is that index information plus some knowledge about the dynamics on N can, in many cases, be used to obtain reasonably strong conclusions concerning the dynamics of S. In particular, Theorem 4.20 states that if S has the Conley index of a periodic orbit and that the isolating neighborhood possesses a Poincaré section, then S must contain a periodic orbit.

To apply Theorem 3.16 one needs to be able to determine the orientability of the unstable manifold of S. For a general nonlinear differential equation, doing this analytically is extremely difficult. For this reason, the index theory is typically computed using $\mathbf{Z}_2$ coefficients. In this case the Thom isomorphism theorem implies the following result.

Theorem 3.19 *Assume that a manifold S is a normally hyperbolic invariant set. Let E be the vector bundle over S defined by the local unstable manifold of S. If E is a rank n bundle, then*

$$CH_k(S; \mathbf{Z}_2) \approx H_{k+n}(S, \mathbf{Z}_2).$$

While these theorems provide nice abstract means of determining the Conley index, in applications the most common approach is to perform a continuation to a simple system.

Example 3.20 Returning to Example 2.3, recall that N is an isolating neighborhood for all $\lambda \in [-1, 1]$. At $\lambda = -1$, $S_{c,-1} = \emptyset$ and therefore, $CH_*(S_{c,\lambda}) \approx 0$.

Example 3.21 For Example 2.8 we wish to compute the index of S_c. Consider the one parameter family of equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= cy + (1-\lambda)(x^2-1)(x+1/2) + \lambda(x-1), \quad c > 0. \end{aligned} \tag{4}$$

N is an isolating neighborhood for all $\lambda \in [0, 1]$. Furthermore, at $\lambda = 0$ we have the equation (2) while at $\lambda = 1$ we have a system for which $\operatorname{Inv} N$ consists of a single hyperbolic fixed point with a one dimensional unstable manifold. Therefore,

$$CH_k(S_c) \approx \begin{cases} \mathbf{Z} & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem is fundamental to many of the most significant applications of the index theory to date.

Theorem 3.22 (Summation Property) *Assume $S = S_0 \cup S_1$ is an isolated invariant set where S_0 and S_1 are disjoint invariant sets. Then*

$$CH_*(S) \approx CH_*(S_0) \oplus CH_*(S_1).$$

Proof. Since S_0 and S_1 are disjoint invariant sets, there exist disjoint isolating neighborhoods N_0 and N_1 such that (N_0, L_0) and (N_1, L_1) are index pairs for S_0 and S_1 respectively. Thus,

$$\begin{aligned} CH_*(S) &= H_*(N_0 \cup N_1, L_0 \cup L_1) \\ &= H_*(N_0, L_0) \oplus H_*(N_1, L_1) \\ &= CH_*(S_0) \oplus CH_*(S_1). \end{aligned}$$

∎

Example 3.23 Consider yet again Example 2.3. The index computations of Examples 3.14 and 3.20 allow us to apply Theorem 3.22 to conclude that $S \neq \{(\pm\lambda, 0)\}$. Therefore, Theorem 2.4 implies that there exists a heteroclinic orbit from $(-\lambda, 0)$ to $(\lambda, 0)$.

3.2 Maps

As was indicated in the introduction the Conley index can be defined for discrete dynamical systems, i.e. for continuous maps $f : X \to X$ where X is a locally compact metric space. As in the case of flows one begins with the notion of an index pair.

Definition 3.24 A pair (N, L) of compact subsets of X is called an *index pair* for an isolated invariant set S if $L \subset N$ and

1. $\mathrm{cl}(N \backslash L)$ is an isolating neighborhood isolating S,
2. $f(L) \cap N \subset L$ (positive invariance),
3. $f(N \backslash L) \subset N$ (exit set).

Theorem 3.25 *For any neighborhood V of an isolated invariant set S there exists an index pair (N, L) for S such that $N \backslash L \subset V$.*

So far the Conley index theory in the discrete case mimics the flow case. To construct the index we need to take two index pairs (N, L) and (N', L') for the same isolated invariant set S and show that they carry some common information. In the flow case the most general common information is the homotopy type of the quotient space $[N/L]$ and the required homotopies are built along the trajectories of the flow. The difficulty in generalizing the Conley index to maps lies in the lack of such homotopies. Actually things are even worse: the homotopies needed in the construction of the Conley index for flows may not exists at all in the discrete case. Consider the following two examples.

Example 3.26 Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) := x + 1$. For every $n \in \mathbf{N}$ put $L_n := [n, n + 1/2]$ and $N_n := \bigcup\{L_i | i = 0, 1, 2, ...n\}$. One can easily verify that for every $n \in \mathbf{N}$ the pair (N_n, L_n) is an index pair for f such that $N_n \backslash L_n$ isolates the empty set. On the other hand $[N_n/L_n, [L_n]]$ is the homotopy type of a pointed set of n elements.

Example 3.27 *(Smale's Horseshoe).* Let $N := [0, 1] \times [0, 1] \subset \mathbf{R}^2$. Assume $f : \mathbf{R}^2 \to \mathbf{R}^2$ is a continuous map such that f maps two rectangles R_0 and R_1 linearly onto rectangles S_0 and S_1, as indicated in Fig. 1. Assume also that f maps $N \backslash (R_0 \cup R_1)$ into $\mathbf{R}^2 \backslash N$ and $\mathbf{R}^2 \backslash N$ into $\mathbf{R}^2 \backslash (S_0 \cup S_1)$.

Put

$$C_n := \{\alpha_1/5 + ... + \alpha_n/5^n + \alpha/5^{n+1} \mid \alpha_i \in \{1, 3\}, i = 1, 2, ...n, \alpha \in [0, 1]\},$$

$$L_n := [0, 1] \times \mathrm{cl}([0, 1] \backslash C_n).$$

Then $S := \mathrm{Inv}\, N \subset \bigcap\{C_n \times C_n | n = 1, 2, ...\}$. One can easily verify that (N, L_n) is an index pair for S for all $n \in \mathbf{N}$ and $[N/L_n, [L_n]]$ is the homotopy type of the wedge sum of 2^n copies of a pointed circle.

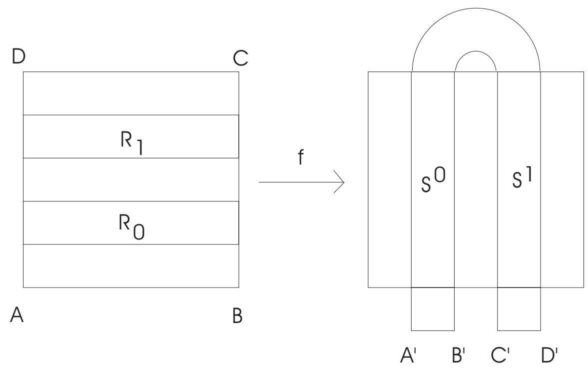

Figure 1: The U horseshoe.

The examples show that unlike the flow case, the homotopy type of the quotient space $[N/L]$ of an index pair (N, L) is not an invariant of S. This can raise the question if the Conley index in the discrete case makes sense at all. Surprisingly it does, and the theory incorporates the index for flows as a special case. To see how this can happen we need to introduce the index map. Given an index pair (N, L), the *index map* is the map $f_{N,L} : N/L \to N/L$ defined by

$$f_{N,L}([x]) := \begin{cases} f(x) & \text{if } f(x) \in N \\ [L] & \text{otherwise.} \end{cases} \tag{5}$$

The following proposition follows easily from the definition of the index pair.

Proposition 3.28 *For any index pair (N, L) the index map $f_{N,L}$ is continuous.*

Index maps turn out to be essential in extracting common information from index pairs of the same isolated invariant set. This may be done in several different ways depending on how general the information we want to extract is. Recently Franks and Richeson [24] proposed to use shift equivalence.

Assume $c : C \to C$ and $d : D \to D$ are continuous maps. They are *shift equivalent* if there exist continuous maps $r : C \to D$, $s : D \to C$ and a natural number m such that $rc = dr$, $sd = cs$, $rs = c^m$ and $sr = d^m$. We say that the homotopy classes of c and d are shift equivalent if there exist continuous maps $r : C \to D$, $s : D \to C$ and a natural number m such that $rc \sim dr$, $sd \sim cs$, $rs \sim c^m$ and $sr \sim d^m$. It is straightforward to verify that shift equivalence is an equivalence relation.

Theorem 3.29 *If (N, L) and (N', L') are index pairs for an isolated invariant set S, then the homotopy classes of index maps $[f_{N,L}]$ and $[f_{N',L'}]$ are shift equivalent.*

We define $h(S, f)$, the *homotopy Conley index* of S as the shift equivalence class of $[f_{N,L}]$. The correctness of this definition is justified by Theorem 3.29.

One can prove that the Conley index for discrete dynamical systems has the same basic properties as the Conley index for flows. In particular it has the Ważewski Property, Continuation Property and Summation Property.

The homotopy Conley index it is the most difficult to compute, because we do not know much about shift equivalence in the homotopy category of compact metric spaces. Therefore in applications we usually work with algebraic Conley indices, which may be obtained from the homotopy index by applying algebraic functors like the homology or cohomology functor.

In the sequel we will study the simplest case when the homology coefficients are in $\mathbf{Q}$ and the phase space X is a compact ANR (for instance a manifold or a polyhedron). Under such assumptions one can show that in every neighborhood of an isolated invariant set there exist index pairs (N, L) such that $H_*(N, L)$ is a finite dimensional vector space.

Two endomorphisms of finite dimensional vector spaces $e : E \to E$ and $f : F \to F$ are *shift equivalent* if there exist linear maps $r : E \to E$, $s : F \to F$ and a natural number m such that $re = fr$, $sf = es$, $rs = e^m$ and $sr = f^m$. Applying the homology functor with rational coefficients to the homotopy Conley index we obtain the *homology Conley index* $\mathrm{Con}_*(S, f)$ as the shift equivalence of $H_*(f_{N,L})$. Such shift equivalences may be found easily by means of the Leray reduction. Let $e : E \to E$ be an endomorphism of a finitely dimensional vector space. The *generalized kernel* of e is defined by

$$\mathrm{gker}(e) := \bigcup\{e^{-n}(0) | n \in \mathbf{N}\}.$$

Since $e(\mathrm{gker}(e)) \subset \mathrm{gker}(e)$, we have an induced isomorphism

$$e' : E/\,\mathrm{gker}(e) \ni [x] \to [e(x)] \in E/\,\mathrm{gker}(e).$$

We call e' the *Leray reduction* of e.

Proposition 3.30 *If e' is the Leray reduction of an endomorphism $e : E \to E$ of a finite dimensional vector space then e and e' are shift equivalent.*

Proposition 3.31 *Assume $e : E \to E$ and $f : F \to F$ are two automorphisms of finite dimensional vector spaces. They are shift equivalent if and only if they are conjugate.*

Since the Leray reduction of a finite dimensional endomorphism is obviously an automorphism, we have the following theorem.

Theorem 3.32 *Two endomorphisms e and f of finite dimensional vector spaces are shift equivalent if and only if their Leray reductions are conjugate.*

By the above theorem we may consider the homological Conley index for maps on compact ANR's as a pair $\mathrm{Con}(S, f) = (CH_*(S, f), \chi_*(S, f))$, where $CH_*(S, f) = \{CH_n(S, f)\}$ is a graded finite dimensional space equal to

$$H_*(N/L, [L])/\,\mathrm{gker}(f_{N,L})_*$$

for any index pair (N,L) such that $H_*(N/L,[L])$ is finite dimensional and $\chi_*(S,f) = \{\chi_n(S,f)\}$ is a graded automorphism on $CH_*(S,f)$ equal to the Leray reduction of $(f_{N,L})_*$.

Consider a hyperbolic fixed point $x_0 \in \mathbf{R}^n$ of a C^1-diffeomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$. Let k denote the number of eigenvalues of $Df(x_0)$ out of the unit circle (counted with multiplicity). Let l denote the number of real eigenvalues of $\mathrm{Df}(x_0)$ which are less than -1. The pair (k,l) will be referred to as the *Morse index* of x_0.

Theorem 3.33 *Assume x_0 is a hyperbolic fixed point of a C^1-diffeomorphism $\mathbf{R}^n \to \mathbf{R}^n$. Then $\{x_0\}$ is an isolated invariant set and*

$$\mathrm{Con}_i(\{x_0\}) = \begin{cases} 0 & \textit{for } i \neq k \\ (\mathbf{Q}, (-1)^l \,\mathrm{id}) & \textit{for } i = k \end{cases} \tag{6}$$

Proof. Without loss of generality we may assume that $x_0 = 0$. We will first consider the case when f is linear, i.e. $f(x) = Ax$ for some $A \in \mathbf{R}^{n\times n}$. Let $\mathbf{R}^n = U \oplus V$ be the decomposition of $\mathbf{R}^n$ into eigenspaces of A corresponding to eigenvalues with modulus greater than one and less than one respectively. Let

$$A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$$

be the corresponding decomposition of A. ¿From the decomposition it follows that $\{0\}$ is the only compact trajectory and consequently it is an isolated invariant set.

In order to prove (6) in the linear case first assume that A has n different eigenvalues. Choose a basis in which A has the block-diagonal matrix with one-dimensional blocks $[\lambda_j]$ corresponding to the real eigenvalues λ_j and two-dimensional blocks

$$\begin{bmatrix} r_j \cos\varphi_j & r_j \sin\varphi_j \\ -r_j \sin\varphi_j & r_j \cos\varphi_j \end{bmatrix}$$

corresponding to the pairs of complex eigenvalues $r_j \exp(-i\varphi_j)$ and $r_j \exp(i\varphi_j)$. For $t \in [0,1]$ let A_t be the matrix of the same block-diagonal structure and with corresponding blocks of the form

$$[h(\lambda_j, t)]$$

and

$$h(r_j,t) \begin{bmatrix} \cos t\varphi_j & \sin t\varphi_j \\ -\sin t\varphi_j & \cos t\varphi_j \end{bmatrix}$$

where

$$h(u,t) := tu + (1-t)\,\mathrm{sgn}(u) \exp(\mathrm{sgn}(|u|-1)\ln 2).$$

Then the origin is the only non-trivial isolated invariant set with respect to each A_t and the homotopy property of the Conley index shows that $\mathrm{Con}(\{0\}, A_1) = \mathrm{Con}(\{0\}, A_0)$. Thus we may assume that $A = A_0$, i.e. A has a diagonal matrix in which -2 appears l times, 2 appears $k-l$ times and the remaining non-zero entries are 1/2 or -1/2.

Put

$$\begin{array}{rcl} N & := & \{x+y \in U \oplus V \times \mathbf{R}^{n-k} \mid ||x|| \leq 2, ||y|| \leq 1\}, \\ L & := & \{x+y \in N \mid ||x|| \geq 1\}, \\ B & := & \{x \in U \mid ||x|| \leq 1\}, \\ S & := & \{x \in U \mid ||x|| = 1\}, \end{array}$$

It is easy to verify that (N, L) is an index pair of $\{0\}$. Let the mappings

$$\begin{array}{rcl} d & : & (B,S) \to (B,S) \\ \alpha & : & (B,S) \to (N,L) \end{array}$$

be given by

$$\begin{array}{rcl} d(u,v) & := & (u,-v), \\ \alpha(x) & := & (x,0). \end{array}$$

One can easily verify that the following diagram

$$\begin{array}{ccc} N/L & \stackrel{f_{N,L}}{\to} & N/L \\ \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \alpha} \\ B/S & \stackrel{d}{\to} & B/S \end{array}$$

is commutative up to a homotopy. Thus we get the following diagram in homology

$$\begin{array}{ccc} H_*(N/L) & \stackrel{(f_{N,L})_*}{\to} & H_*(N/L) \\ \big\downarrow{\scriptstyle \alpha_*} & & \big\downarrow{\scriptstyle \alpha*} \\ H_*(B/S) & \stackrel{d_*}{\to} & H_*(B/S). \end{array}$$

It is straightforward to verify that α_* and d_* are isomorphisms. Therefore

$$\mathrm{Con}(\{0\}, f) = (H_*(B/S), d_*) = \begin{cases} 0 & \text{for } i \neq k \\ (\mathbf{Q}, (-1)^l \,\mathrm{id}) & \text{for } i = k. \end{cases}$$

In order to obtain formula (6) in case of a linear map with multiple eigenvalues it suffices to construct a homotopy joining the map A with a nearby map A' which has pairwise different eigenvalues and then apply the homotopy property.

Now we shall consider the case of a general f. The fact that also in this case $\{0\}$ is an isolated invariant set follows directly from the linear case and the Hartman-Grobman theorem. Let $A := Df(0)$. We have

$$f(x) = Ax + r(x),$$

where $r(x) = o(||x||)$. For $\lambda \in [0,1]$ define

$$f_\lambda(x) := Ax + \lambda r(x).$$

For each fixed $\lambda \in [0,1]$ we can apply the Hartman-Grobman theorem to find $\delta(\lambda) > 0$ such that $B(\delta(\lambda)) := \{x \in \mathbf{R}^n : ||x|| \leq \delta(\lambda)\}$ is an isolating neighborhood with respect to f_λ, isolating $\{0\}$. A compactness argument shows that there exists a $\delta > 0$ such that $B(\delta)$ is an isolating neighborhood with respect to f_λ for each $\lambda \in [0,1]$. Obviously we can make δ small enough to ensure that $\mathrm{Inv}(B(\delta), f) = \{0\}$. Since $f_1 = f$, $f_0 = A$, the thesis follows now from the homotopy invariance of the Conley index and the proved linear case. ∎

Theorem 3.34 *Let $x_0 \in \mathbf{R}^n$ be a hyperbolic periodic orbit of f, i.e. a hyperbolic fixed point of f^d for some $d \in \mathbf{N}$. Let (k,l) be the Morse index of x_0 with respect to f^d. Assume d is the minimal period of x_0, i.e. $f^i(x_0) \neq x_0$ for $i = 1, 2, ...d-1$. Then $S := \{x_0, f(x_0), ..., f^{d-1}(x_0)\}$ is an isolated invariant set and*

$$\mathrm{Con}_i(S) = \begin{cases} 0 & \text{for } i \neq k \\ (\mathbf{Q}^d, D) & \text{for } i = k, \end{cases} \tag{7}$$

where $D : \mathbf{Q}^d \to \mathbf{Q}^d$ is given on the canonical basis $\{e_i\}_{i=1,d}$ of $\mathbf{Q}^d$ by

$$D(e_i) = e_{i+1} \text{ for } i = 1, 2, ..., d-1,$$

$$D(e_d) = (-1)^l e_1.$$

Proof. Choose (N, L), an index pair of $\{x_0\}$ with respect to $g := f^d$. Put $x^i := f^i(x_0)$, $N^i := f^i(N), L^i := f^i(L)$ for $i = 0, 1, ..., d-1$. Taking N smaller if necessary, we may assume that $N^i \cap N^j = \emptyset$ for $i \neq j$ and $f(N^{d-1}) \cap N = \emptyset$.

Obviously x^i are hyperbolic fixed points of g with the same Morse index (k, l). One can easily verify that (N^i, L^i) for $i = 0, 1, ..., d-1$ is an index pair for g. It follows that $N' := \bigcup_{i=0}^{d-1} N^i$ is an isolating neighborhood with respect to f and (N', L') with $L' := \bigcup_{i=0}^{d-1} L^i$ is an index pair for f which isolates $\{x^0, x^1, ..., x^{d-1}\}$. We have

$$H_*(N'/L') = H_*(N^0/L^0) \oplus ... \oplus H_*(N^{d-1}/L^{d-1}).$$

Hence it follows from Theorem 3.33 applied to g that $H_i(N'/L')$ is zero for $i \neq k$ and $H_k(N'/L')$ is a d-dimensional vector space.

Choose arbitrarily a generator $\alpha^0 \in H_k(N^0/L^0)$ and define recursively a sequence $\{\alpha^i\}_{i=1,d-1}$ of generators in $H_k(N^i/L^i)$ by $\alpha^{i+1} := H_k(f)(\alpha^i)$ for $i = 0, 1, ..., d-2$. Then $\{\epsilon^i\}_{i=0,d-1}$ with $\epsilon^i := (0, ..., \alpha^i, ...0)$ is a basis of $H_k(N'/L')$. It is straightforward to verify that $(f_{N',L'})_k(\epsilon^i) = \epsilon^{i+1}$ for $i = 0, 1, 2, ..., d-2$. We have also by (6) that

$$(f_{N',L'})_k(\epsilon^{d-1}) = (f^d_{N',L'})_k(\epsilon^0) = (g_{N',L'})_k(\alpha^0, 0, ..., 0) =$$

$$= ((-1)^l \alpha^0, 0, ..., 0) = (-1)^l \epsilon^0.$$

This shows that $(f_{N',L'})_*$ is an isomorphism and formula (7) holds. ∎

Example 3.35 Consider again the horseshoe map from Example 3.27. Take

$$L := [0,1] \times ([0, \frac{1}{5}] \cup [\frac{2}{5}, \frac{3}{5}] \cup [\frac{4}{5}, 1]).$$

Then (N, L) is an index pair for f and $(f_{N,L})_*$ has the matrix

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

It follows that $(f_{N,L})_*^2 = 0$ and consequently the homological Conley index of the classical Smale's horseshoe is zero.

Example 3.36 This is not the case for the isolated invariant set S of the horseshoe in Figure 2. (N, L) as above is again an index pair but $(f_{N,L})_*$ has the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

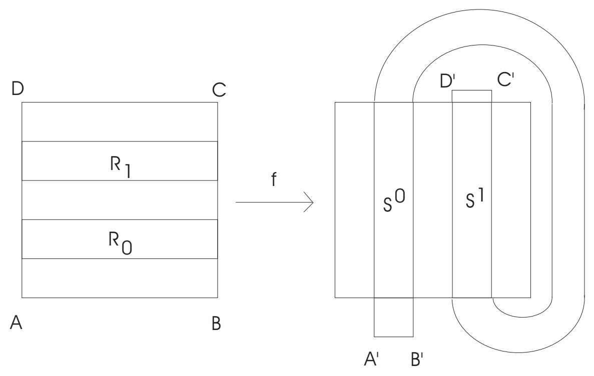

Figure 2: The G horseshoe.

Since in this case $(f_{N,L})_*^2 = 2(f_{N,L})_*$, it follows that

$$\mathrm{Con}_k(S) = \begin{cases} 0 & \text{for } k \neq 1, \\ (\mathbf{Q}, 2\,\mathrm{id}) & \text{for } k = 1. \end{cases}$$

3.3 References

The original development of the Conley index for flows took place for the most part in the seventies. Again [7] and [75] are excellent sources. Early work includes that of Conley and R. Easton on isolating blocks [9]. Since that time the theory has undergone several developments. One important direction is the generalization to semi-flows on arbitrary metric spaces. For this one should consult the work of K. Rybakowski [74], V. Benci [2] and V. Benci and M. Degiovanni [3].

The first version of the Conley index for maps was presented by Robbin and Salamon [70]. Their definition of an index pair is more general than that presented here in that they only require that the compact pair (N, L) posses the property that the induced index map $f_{N,L} : N/L \to N/L$ be continuous. The index based on the Leray reduction was proposed in [62]. In this paper it was also observed that the index map gives rise to an index automorphism which is also an invariant of the isolated invariant set. Szymczak [78] constructed the homotopy Conley index as a functor into an abstract category and showed that any other Conley index can be factorized through his homotopy Conley index. Recently Franks and Richeson [24] presented a version of Szymczak's approach based on shift equivalences.

4 The Index and the Structure of Invariant Sets

4.1 The Structure of Morse Decompositions

In order to fully describe the dynamics of an isolated invariant set for which we have a Morse decomposition we need to understand the connecting orbits between the Morse sets. This will be done by relating the Conley indices of Morse sets to the Conley index of the total isolated invariant set. The construction of the algebraic machinery needed to do this in the general setting of Morse decompositions is rather formidable and will not be presented in these notes. However, to give the reader a flavor of the issues involved we shall indicate in some detail what happens in the setting of an attractor repeller pair decomposition. For the sake of simplicity we will restrict our discussion to the setting of flows.

4.1.1 Attractor Repeller Pairs

The goal of this section is to show that it is possible to reduce the answer to the question of the existence of a connecting orbit in Example 2.6 to a problem in linear algebra. More precisely we will show that given an isolated invariant set S with an attractor-repeller pair decomposition (A, R), the relationship between their Conley indices can be expressed in terms of a matrix called the connection matrix. The first step is to be able to determine the indices in a consistent manner.

Definition 4.1 Let S be an isolated invariant set and let (A, R) be an attractor–repeller pair decomposition. An *index triple* for (A, R) is a collection of compact sets (N_2, N_1, N_0) where $N_0 \subset N_1 \subset N_2$ such that:

1. (N_2, N_0) is an index pair for S;
2. (N_2, N_1) is an index pair for R;
3. (N_1, N_0) is an index pair for A.

Theorem 4.2 *Let (A, R) be an attractor–repeller pair decomposition of an isolated invariant set S. Then, there exists an index triple (N_2, N_1, N_0).*

Proof. By Theorem 3.5 there exists an index pair (N_2, N_0) for S. It is left to the reader to check that there exists a closed neighborhood U of A and a constant $\epsilon > 0$ such that if $x \in U$ and $\varphi([0,t], x) \subset N_2$ then $\mu(\varphi([0,t], x), R) > \epsilon$. Define

$$Z := \{y \in N_2 \mid \exists\, t > 0\ x \in U \text{ such that } \varphi([0,t], x) \subset N_2 \text{ and } \varphi(t, x) = y\}.$$

Let $N_1 := N_0 \cup Z$. Then (N_2, N_1, N_0) is the desired index triple. ∎

Since $N_0 \subset N_1 \subset N_2$ there exists on the chain level a short exact sequence

$$0 \to C_*(N_1, N_0) \to C_*(N_2, N_0) \to C_*(N_1, N_0) \to 0$$

which gives rise to the long exact sequence

$$\ldots \to H_n(N_1, N_0) \to H_n(N_2, N_0) \to H_n(N_1, N_0) \xrightarrow{\partial_n} H_{n-1}(N_2, N_1) \to \ldots \quad (8)$$

But this is equivalent to

$$\ldots \to CH_n(A) \to CH_n(S) \to CH_n(R) \xrightarrow{\partial_n} CH_{n-1}(A) \to \ldots \quad (9)$$

The following theorem relates this exact sequence to the underlying dynamics.

Theorem 4.3 *Let (A, R) be an attractor–repeller pair decomposition of an isolated invariant set S. If $S = A \cup R$, then $\partial_n = 0$.*

Proof. Let U and V be disjoint neighborhoods of A and R, respectively. Since $S = A \cup R$ there exists an index triple of the form

$$(N_1^A \cup N_2^R, N_1^A \cup N_0^R, N_0^A \cup N_0^R)$$

where $N_i^A \subset U$ and $N_i^R \subset V$. Equation (8) now takes the form

$$\begin{aligned} \ldots &\to H_n(N_1^A \cup N_0^R, N_0^A \cup N_0^R) \to H_n(N_1^A \cup N_2^R, N_0^A \cup N_0^R) \\ &\to H_n(N_1^A \cup N_2^R, N_1^A \cup N_0^R) \xrightarrow{\partial_n} H_{n-1}(N_1^A \cup N_0^R, N_0^A \cup N_0^R) \to \ldots \end{aligned}$$

which by excision is equivalent to

$$\begin{aligned} \ldots &\to H_n(N_2^R, N_0^R) \to H_n(N_1^A, N_0^A) \oplus H_n(N_2^R, N_0^R) \\ &\to H_n(N_2^R, N_0^R) \xrightarrow{\partial_n} H_{n-1}(N_1^R, N_0^A) \to \ldots \end{aligned}$$

Since the sequence splits, $\partial_n = 0$. ∎

It is the contrapositive of this theorem which is typically of use in applications.

Corollary 4.4 *If $\partial_n(A, R) \neq 0$, then $S \neq A \cup R$, i.e $C(R, A; S) \neq \emptyset$.*

Remark 4.5 Example 3.18 shows that the converse of this corollary is not true.

In preparation for the rather complicated algebra which will be introduced in 4.1.2, we will now try to present the information that is carried by ∂_* in the form of a matrix. To simplify the presentation we shall now only consider homology with *field* coefficients. The information which we have consists of three parts.

1. A collection of graded vector spaces arising from the indices of the attractor and repeller,
$$CH_*(A) \oplus CH_*(R).$$
2. The index of the total invariant set S,
$$CH_*(S).$$
3. The *connecting homomorphism* or *boundary map* ∂_n which is a degree -1 operator, i.e. it sends n-level homology to $n-1$-level homology. We shall, from now on, denote this homomorphism by
$$\partial_*(A,R) : CH_*(R) \to CH_*(A).$$

We now pose the following question. Let us view $CH_*(A) \oplus CH_*(R)$ as a chain complex with a boundary operator Δ. Can we choose Δ in such a way that the resulting homology $H\Delta_*$ is isomorphic to $CH_*(S)$? To be more precise, let

$$\Delta_n : CH_n(A) \oplus CH_n(R) \to CH_{n-1}(A) \oplus CH_{n-1}(R).$$

denote the obvious restriction of Δ to each homology level of the indices. Then, by definition

$$H\Delta_n := \frac{\ker(\Delta_n)}{\text{image}(\Delta_{n+1})}.$$

Thus we are asking whether it is possible to choose Δ in such a way that

$$H\Delta_n \approx CH_n(S) \quad n = 0, 1, 2, \ldots$$

Observe that if $S = A \cup R$, then $CH_*(S) \approx CH_*(A) \oplus CH_*(R)$. In this case we can choose $\Delta = 0$. This is a special case of the following proposition.

Proposition 4.6 *If we define*

$$\Delta = \begin{bmatrix} 0 & \partial_n(A,R) \\ 0 & 0 \end{bmatrix} : CH_*(A) \oplus CH_*(R) \to CH_*(A) \oplus CH_*(R)$$

then

$$H\Delta_* \approx CH_*(S).$$

Proof. We begin by checking that Δ is a boundary operator, i.e. that Δ is degree -1 and $\Delta \circ \Delta = 0$. The first is obvious since ∂_n is of degree -1. The second condition is equally obvious.

We now need to show that

$$H\Delta_n \approx CH_n(S) \quad n = 0, 1, 2, \ldots$$

Observe that

$$\ker \Delta_n = CH_n(R) \oplus \ker \partial_n(A, R)$$

and

$$\text{image}\, \Delta_{n-1} = \text{image}\, \partial_{n-1}(A, R).$$

Thus, we need to show that

$$\begin{aligned} CH_n(S) &\approx \frac{CH_n(A) \oplus \ker \partial_n(A, R)}{\text{image}\, \partial_{n+1}(A, R)} \\ &\approx \frac{CH_n(A)}{\text{image}\, \partial_{n+1}(A, R)} \oplus \ker \partial_n(A, R). \end{aligned}$$

Now consider the sequence

$$\ldots \to CH_n(A) \stackrel{i_n}{\to} CH_n(S) \stackrel{j_n}{\to} CH_n(R) \stackrel{\partial_n}{\to} CH_{n-1}(A) \to \ldots$$

which, since we are working with field coefficients, can be written as

$$\ldots \to CH_n(A) \to \text{image}\, i_n \oplus L_n \to CH_n(R) \stackrel{\partial_n}{\to} CH_{n-1}(A) \to \ldots$$

By exactness

$$L_n \approx \ker \delta_n(A, R))$$

and

$$\text{image}\, i_n \approx \frac{CH_n(A)}{\text{image}\, \partial_{n+1}(A, R)}.$$

∎

The matrix in Proposition 4.6 is the simplest example of a connection matrix which will be described further in the next section.

Example 4.7 We now return to Example 2.3 and show that for all $c > 0$ and $0 < \lambda \leq 1$, there exists a connecting orbit from $(-\sqrt{\lambda}, 0)$ to $(\sqrt{\lambda}, 0)$. For the index computations we shall use the field coefficients $\mathbf{Z}_2$. In Example 3.20 it was shown that $CH_*(S_{c,\lambda}) = 0$. Linearizing about the equilibria shows that

$$CH_k(A) \approx \begin{cases} \mathbf{Z_2} & \text{if } k = 1 \\ 0 & \text{otherwise,} \end{cases} \quad CH_k(R) \approx \begin{cases} \mathbf{Z_2} & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the portion of the exact sequence (9) which is of interest is

$$\ldots \to CH_2(S) \to CH_2(R) \stackrel{\partial_2}{\to} CH_1(A) \to CH_1(S) \to \ldots$$

which of course reduces to

$$\ldots \to 0 \to \mathbf{Z}_2 \overset{\partial_2}{\to} \mathbf{Z}_2 \to 0 \to \ldots$$

Since this sequence is exact $\partial_2(A,R) \neq 0$. By Corollary 4.4 there exists a heteroclinic orbit from R to A.

We can abstract this example into the following proposition.

Proposition 4.8 *Assume that S is an isolated invariant set with an attractor–repeller pair decomposition (A,R) consisting of two fixed points. Furthermore, assume that*

$$CH_n(R) \approx \begin{cases} \mathbf{Z} & \text{if } n = k+1 \\ 0 & \text{otherwise,} \end{cases} \qquad CH_n(A) \approx \begin{cases} \mathbf{Z} & \text{if } n = k \\ 0 & \text{otherwise,} \end{cases}$$

and

$$CH_*(S) \approx 0.$$

Then, there exists a heteroclinic orbit from R to A.

Proof. By Theorem 3.22, $S \neq A \cup R$. Thus, by Theorem 2.4, $C(R,A;S) \neq \emptyset$

∎

Consider Proposition 4.8. This is a pure existence result; the conclusion is that there exists an orbit $\varphi(\mathbf{R},x) \subset S$ such that

$$\lim_{t\to\infty} \varphi(t,x) = A \qquad \lim_{t\to-\infty} \varphi(t,x) = R.$$

What it does not do is describe the dynamics on S as a whole. Clearly, in this simple example it is obvious that the only type of orbits which can occur are the fixed points A and R and the heteroclinic orbits from R to A.

However, for the sake of clarity it is worthwhile trying to recast this example in the context of semi–conjugacies. More complicated examples will be presented in later sections, but there the technical difficulties can overwhelm the simple idea. With this in mind, therefore, we shall prove the following theorem.

Theorem 4.9 *Let φ be a flow for which S is an isolated invariant set with an attractor–repeller pair decomposition (A,R). Assume that*

$$CH_*(S) \not\approx CH_*(A) \oplus CH_*(R).$$

Let $\psi : \mathbf{R} \times [-1,1] \to [-1,1]$ be the flow on the unit interval generated by

$$\dot{x} = (x^2 - 1).$$

Then, there exists a continuous surjective function $\rho : S \to [-1,1]$ such that

$$A = \rho^{-1}(-1) \quad R = \rho^{-1}(1)$$

and the following diagram commutes

$$
\begin{tikzcd}
\mathbf{R} \times S \arrow[r, "\varphi"] \arrow[d, "id \times \rho"'] & S \arrow[d, "\rho"] \\
\mathbf{R} \times [-1,1] \arrow[r, "\psi"] & [-1,1]
\end{tikzcd}
$$

Proof. Let $V : S \to [0,1]$ be a Lyapunov function for φ as defined by Theorem 2.7. Because V is strictly decreasing off of $A \cup R$, given $x \in C(R, A; S)$ there exists a unique $t_x \in \mathbf{R}$ such that

$$V(\varphi(t_x, x)) = \frac{1}{2}.$$

Observe that the function $\tau : C(R, A; S) \to \mathbf{R}$ given by $\tau(x) = t_x$ is continuous. Now, define $\rho : S \to [0,1]$ by

$$\rho(x) = \begin{cases} -1 & \text{if } x \in A \\ 1 & \text{if } x \in R \\ -\frac{2}{\pi} \tan^{-1} t_x & \text{if } x \in C(R, A; S) \end{cases}.$$

Since $CH_*(S) \not\approx CH_*(A) \oplus CH_*(R)$, $C(R, A; S) \neq \emptyset$, and hence, ρ is surjective. ∎

There are several points that should be remarked concerning the proof of this theorem. The first is that one uses the decomposition and the Lyapunov function to define the semi-conjugacy. The second is that it is the Conley index information that allows one to conclude that ρ is a surjection.

On a more general level, the reader should note that a theorem of this form provides a global description of the dynamics. The existence of a semi–conjugacy provides a lower bound on the complexity of the dynamics. Since ρ is surjective, every trajectory of ψ is lifted to ϕ. Of course this lifting need not be one to one, and hence, the dynamics of ϕ may be more complicated than that of ψ.

Just as one can decompose isolated invariant sets, one can decompose sets of connecting orbits. Let (A, R) be an attractor-repeller pair decomposition of S with associated boundary operator $\partial(A, R)$. A *separation* of $C(R, A)$ is a collection $\{C_j(R, A) \mid j = 1, \ldots, J\}$ of open subsets of $C(R, A)$ such that

$$C(R, A) = \bigcup_{j=1}^{J} C_j(R, A).$$

If N is an isolating neighborhood for S, then there exists $N_j \subset N$ such that N_j is an isolating neighborhood for $S_j := A \cup R \cup C_j(R, A)$. Observe that (A, R) is an attractor-repeller pair for S_j, and hence, there is an associated boundary operator $\partial(A, R; j)$.

Theorem 4.10 (McCord[45]) *For any separation of* $C(R, A)$

$$\partial(A, R) = \sum_{j=1}^{J} \partial(A, R; j).$$

4.1.2 Connection Matrices

In the previous section we constructed a connection matrix for the case of an attractor repeller pair. We now wish to do the same for a Morse decomposition. As was indicated earlier, the construction is not trivial. For this reason we will only describe the properties that connection matrices posses, referring the reader to [25, 26, 27, 28, 69, 71] for further details. We remind the reader that throughout this section we will be using homology with *field* coefficients. In practice one usually uses $\mathbf{Z}_2$ coefficients since this avoids one having to deal with orientations.

Let $\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}$ be a Morse decomposition. Each Morse set is an isolated invariant set, and therefore, has a Conley index of the form $CH_*(M(p))$. Let

$$\Delta : \bigoplus_{p \in \mathcal{P}} CH_*(M(p)) \to \bigoplus_{p \in \mathcal{P}} CH_*(M(p))$$

be a linear operator which will be written as a matrix

$$\Delta = [\Delta(p, q)]$$

where $\Delta(p, q) : CH_*(M(q)) \to CH_*(M(p))$. Δ is *upper triangular* with respect to the partial order $>$ if $p \not> q$ implies $\Delta(q, p) = 0$. Δ is a *boundary operator* if $\Delta \circ \Delta = 0$ and it is a *degree* -1 *map*, i.e. it maps n^{th} level homology to $n - 1^{st}$ level homology. Finally, for each interval $I \in \mathcal{I}(\mathcal{P}, >)$ define

$$\Delta(I) = [\Delta(p, q)]_{p,q \in I} : \bigoplus_{p \in I} CH_*(M(p)) \to \bigoplus_{p \in I} CH_*(M(p))$$

If Δ is a *connection matrix*, then it satisfies the following four properties:

1. Δ is upper triangular.
2. Δ is a boundary operator.
3. For every interval $I \in \mathcal{I}(\mathcal{P}, >)$
$$H_*\Delta(I) := \frac{\ker \Delta(I)}{\operatorname{image} \Delta(I)} \approx CH_*(M(I))$$
4. Consider a pair of adjacent intervals $(I, J) \in \mathcal{I}_2(\mathcal{P}, >)$. Observe that there are three associated chain complexes: (1) $\bigoplus_{p \in I} CH_*(M(p))$ with boundary operator $\Delta(I)$, (2) $\bigoplus_{p \in J} CH_*(M(p))$ with boundary operator $\Delta(J)$,

and (3) $\bigoplus_{p \in IJ} CH_*(M(p))$ with boundary operator $\Delta(IJ)$. Furthermore, under inclusion and projection the following is a short exact sequence,

$$0 \to \bigoplus_{p \in I} CH_*(M(p)) \to \bigoplus_{p \in IJ} CH_*(M(p)) \to \bigoplus_{p \in J} CH_*(M(p)) \to 0.$$

Applying the boundary operators one obtains the following long exact sequence

$$\cdots \to H_n\Delta(I) \to H_n\Delta(IJ) \to H_n\Delta(J) \to H_{n-1}\Delta(I) \to \cdots$$

Then, this sequence is isomorphic to the long exact sequence for the attractor repeller pair $(M(I), M(J))$ decomposition of $M(IJ)$

$$\cdots \to CH_n(M(I)) \to CH_n(M(IJ)) \to CH_n(M(J)) \to CH_{n-1}(M(I)) \to \cdots$$

From property (4) one can deduce that given adjacent Morse sets $M(p)$ and $M(q)$, i.e. a pair of Morse sets such that $(M(q), M(p)) \in \mathcal{I}_2(\mathcal{P}, >)$, then

$$\Delta(qp) = \begin{bmatrix} 0 & \partial_*(M(q), M(p)) \\ 0 & 0 \end{bmatrix}.$$

It is important to mention that the four properties listed above do not define the connection matrix. For the proper definition the reader is referred to [26]. The fundamental theorem is the following.

Theorem 4.11 (Franzosa [26]) *For any isolated invariant set S with Morse decomposition $\mathcal{M}(S)$ the set of connection matrices are not empty.*

As is implied in the theorem, connection matrices need not be unique.

Example 4.12 Consider once again Example 2.8. We shall use $\mathbf{Z}_2$ coefficients for the index computations. The associated connection matrices must be linear operators of the form

$$\Delta : \bigoplus_{p=1}^{3} CH_*(M(p)) \to \bigoplus_{p=1}^{3} CH_*(M(p)).$$

From Theorem 3.13 each of these is a one-dimensional vector space implying that Δ is a 3×3 matrix. Using the upper triangularity with respect to $>$ and the fact that Δ is a degree -1 map one obtains:

$$\Delta = \begin{bmatrix} 0 & 0 & ? \\ 0 & 0 & ? \\ 0 & 0 & 0 \end{bmatrix}.$$

Now consider the case of $0 < c << 1$. As was indicated in Example 2.10 the flow defined order is $3 >_0 2$ and 1 is unrelated to 2 or 3. Thus upper triangularity with respect to $>_0$ gives

$$\Delta_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & ? \\ 0 & 0 & 0 \end{bmatrix}.$$

In Example 3.21 it was shown that $CH_*(S)$ is one dimensional. Thus applying property 3 of the connection matrices to the interval $\{3,2,1\}$ forces

$$\Delta_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

For the case $c \approx \infty$ the unknown entries are computed directly using property 4. In particular, one considers the attractor–repeller pairs $(M(2), M(3))$ and $(M(1), M(3))$ and shows that $CH_*(M(23)) \approx 0$ and $CH_*(M(13)) \approx 0$. This implies that

$$\Delta_\infty = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

4.1.3 Example - Bistable Gradient Like Systems

In Section 4.1.1 the Conley index was used to obtain a semi-conjugacy describing the dynamics of an attractor repeller pair decomposition for which a connecting orbit exists. One can ask whether similar results can be obtain in the context of Morse decompositions. The answer is yes as the following example indicates.

Consider the following assumptions:

A1 *$\mathcal{A}$ is a global compact attractor for a semi-flow Φ on a Banach space. Furthermore, if φ denotes the restriction of Φ to $\mathcal{A}$ then φ defines a flow on $\mathcal{A}$.*

A2 *Under the flow $\varphi : \mathbf{R} \times \mathcal{A} \to \mathcal{A}$*

$$\mathcal{M}(\mathcal{A}) = \{M(p^\pm) \mid p = 0, \ldots, P-1\} \bigcup \{M(P)\}$$

with ordering $P > P-1^\pm > \ldots > 1^\pm > 0^\pm$ is a Morse decomposition of $\mathcal{A}$.

A3 *The Conley indices of the Morse sets are*

$$CH_k(M(P))) \approx \begin{cases} \mathbf{Z}_2 & \text{if } k = P \\ 0 & \text{otherwise;} \end{cases}$$

and for $p = 0, \ldots, P-1$

$$CH_k(M(p^\pm)) \approx \begin{cases} \mathbf{Z}_2 & \text{if } k = p \\ 0 & \text{otherwise.} \end{cases}$$

A4 *The connection matrix for $\mathcal{M}(\mathcal{A})$ is given by*

$$\Delta = \begin{bmatrix} 0 & D_1 & 0 & \ldots & 0 \\ 0 & 0 & D_2 & & \vdots \\ & 0 & & \ddots & 0 \\ \vdots & & & 0 & D_P \\ 0 & & \ldots & 0 & 0 \end{bmatrix}$$

where for $p = 0, \ldots, P-1$

$$D_p : CH_p(M(p^-)) \oplus CH_p(M(p^+)) \to CH_{p-1}(M(p-1^-)) \oplus CH_{p-1}(M(p-1^+))$$

is given by

$$D_p = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$D_P : CH_P(M(P)) \to CH_{P-1}(M(P-1^-)) \oplus CH_{P-1}(M(P-1^+))$$

is given by

$$D_P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

These assumptions may seem somewhat strange at first, but in fact are satisfied by a wide variety of systems [52]. **A1** is typical of many dissipative systems [37, 36, 81]. The simplest interpretation of assumptions **A2** and **A3** is that the Morse sets consist of hyperbolic fixed points, with the equilibria on the p^{th} level having a p dimensional unstable manifold. Finally, **A4** is a statement that each Morse set on the p^{th} level connects to every Morse set on the $p-1^{\text{st}}$ level.

The simplest ordinary differential equation that posses these properties is the following. Let

$$D^P := \{z = (z_0, \ldots, z_{P-1}) \mid \|z\| \leq 1\} \subset \mathbf{R}^P$$

be the closed unit ball in $\mathbf{R}^P$ and $S^{P-1} = \partial D^P$ be the unit sphere. Let $\psi^P : \mathbf{R} \times D^P \to D^P$ denote the flow generated by the following system of ordinary differential equations

$$\begin{aligned} \dot{\zeta} &= Q\zeta - \langle Q\zeta, \zeta\rangle\zeta \quad \zeta \in S^{P-1} \qquad (10) \\ \dot{r} &= r(1-r) \qquad\qquad r \in [0,1] \qquad (11) \end{aligned}$$

where

$$Q = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & \frac{1}{2} & & \\ \vdots & & \ddots & \\ 0 & & & \frac{1}{P} \end{bmatrix}$$

The dynamics of ψ^P is easily understood if one realizes that (10) is obtained by projecting the linear system $\dot{z} = Qz$ onto the unit sphere. Let $\mathbf{e}_p^{\pm} = (0, \ldots, \pm 1, \ldots, 0)$ be the unit vectors in the p^{th} direction. Then one has the following result.

Theorem 4.13 *Given assumptions* **A1** - **A4** *there exists a continuous surjective map* $f : \mathcal{A} \to D^P$ *and* $\tilde{\varphi}$ *a flow obtained by an order preserving time reparameterization of* φ *such that the following diagram commutes*

$$
\begin{tikzcd}
\mathbf{R} \times \mathcal{A} \arrow[r, "\tilde{\varphi}"] \arrow[d, "id \times f"'] & \mathcal{A} \arrow[d, "f"] \\
\mathbf{R} \times D^P \arrow[r, "\psi^P"] & D^P
\end{tikzcd}
$$

where $M(p^{\pm}) = f^{-1}(\mathbf{e}_p^{\pm})$ *for* $0 \leq p \leq P-1$*, and* $M(P) = f^{-1}(0)$.

The proof of this theorem can be found in [52]. However, the essential idea mimics that of Theorem 4.9. One defines a continuous map from $\mathcal{A}$ to D^P by describing each point in $\mathcal{A}$ in terms of the amount of time its trajectory spends in the neighborhood of the p^{th} Morse set and its value under the abstract Lyapunov function of Theorem 2.11. One then uses the index theory to prove that the map must be surjective.

To apply this theorem the connection matrix must be known. The direct computation of a connection matrix is typically a difficult task and often done through a continuation argument. For additional examples of theorems of this nature and their applications to differential equations the reader is referred to [17, 31, 33, 47, 54, 43]

4.1.4 Transition Matrices

In Examples 2.10 and 4.12 it was shown that the flow defined orders and connection matrices differ for different values of c. This implies the existence of bifurcations. Since the Morse decomposition continues over the parameter range $c > 0$, the bifurcation must be of a global nature. In this section we will introduce the notion of a transition matrix and show that it can be used to detect global bifurcations.

To discuss transition matrices one must begin with a continuous family of flows. We shall take them to be generated by ordinary differential equations defined on $\mathbf{R}^n$,

$$\dot{x} = f(x, \lambda) \quad \lambda \in \mathbf{R}. \tag{12}$$

For the sake of clarity assume that N is an isolating neighborhood for all values of λ and let $S_\lambda = \operatorname{Inv} N$ under the dynamics induced at the parameter value λ. Also, assume that the Morse decomposition $\mathcal{M}(S_\lambda) := \{M_\lambda(p) \mid p \in (\mathcal{P}, >)\}$ also continues. Then choosing $\lambda = \pm 1$ one has two connection matrices $\Delta_\pm$ which are boundary operators on the complexes $\bigoplus CH_*(M_{\pm 1}(p))$ respectively. Transition matrices are chain maps between these chain complexes.

There are several ways in which transition matrices have been defined. The simplest and most general is based on a singular perturbation. Referred to as singular transition matrices, the idea was due to Conley and was carried out by Reineck [68]. Consider the following augmented family of differential equations

$$
\begin{aligned}
\dot{x} &= f(x, \lambda) \\
\dot{\lambda} &= \epsilon(\lambda^2 - 1) \quad \epsilon > 0.
\end{aligned} \tag{13}
$$

For each $\epsilon > 0$ this generates a flow for which $N \times [-2,2]$ is an isolating neighborhood. Let $K_\epsilon = \mathrm{Inv}(N \times [-2,2])$ under the ϵ flow. Define

$$\begin{aligned} M(p^+) &:= M_1(p) \\ M(p^-) &:= M_{-1}(p) \end{aligned}$$

then

$$\mathcal{M}(K_\epsilon) = \{M(p^\pm) \mid p \in \mathcal{P}\}$$

is a Morse decomposition. This follows from the fact that $\epsilon > 0$, and hence, $\dot\lambda < 0$, if $\lambda \in (-1,1)$. Furthermore, an admissible order is given by

$$\begin{array}{lllll} q^+ & >> & p^- & & \\ q^- & >> & p^- & \Leftrightarrow & q > p \\ q^+ & >> & p^+ & \Leftrightarrow & q > p. \end{array}$$

Let Δ_ϵ denote a connection matrix for $\mathcal{M}(K_\epsilon)$. Since the dynamics on the subspaces defined by $\lambda = \pm 1$ are given exactly by the flows generated by $\dot x = f(x, \pm 1)$,

$$\Delta : \bigoplus_{p\in\mathcal{P}} CH_*(M(p^-)) \bigoplus_{p\in\mathcal{P}} CH_*(M(p^+)) \to \bigoplus_{p\in\mathcal{P}} CH_*(M(p^-)) \bigoplus_{p\in\mathcal{P}} CH_*(M(p^+))$$

takes the form

$$\Delta_\epsilon = \begin{bmatrix} \Delta_- & T_\epsilon \\ 0 & \Delta_+ \end{bmatrix} \tag{14}$$

where

$$T_\epsilon : \bigoplus_{p\in\mathcal{P}} CH_{*+1}(M(p^+)) \to \bigoplus_{p\in\mathcal{P}} CH_*(M(p^-))$$

and

$$T_\epsilon(q^-, p^+) : CH_{*+1}(M(p^+)) \to CH_*(M(q^-)).$$

Two comments need to be made at this point. The first is that technically the entries $\Delta_\pm$ in Δ_ϵ cannot be the connection matrices $\Delta_\pm$ for $\mathcal{M}(S_\pm)$, since they are defined on different spaces. One of the contributions of [68] was to overcome this technical obstacle and justify the expression in (14).

Second, while in principle T_ϵ may change as a function of ϵ, it is always possible to choose a sequence $\epsilon_n \to 0$ along which T_{ϵ_n} is constant. The set of matrices obtained through the above mentioned limiting process are referred to as *singular transition matrices* and the set of singular transition matrices is denoted by $\mathcal{T}^{\mathrm{sing}}_{-1,1}$.

Consider $T \in \mathcal{T}^{\mathrm{sing}}_{-1,1}$ and assume p, q are adjacent elements with respect to the ordering $>$ for which $T(q^-, p^+) \neq 0$. Then, there exists a sequence $\epsilon_n \to 0$ for which $T_{\epsilon_n}(q^-, p^+) \neq 0$. This in turn implies the existence of a connecting orbit Γ_{ϵ_n} from $M_1(p)$ to $M_{-1}(q)$ for the parameter value ϵ_n. Taking the limit of Γ_{ϵ_n} as $\epsilon_n \to 0$ implies the existence of a connecting orbit $M_\lambda(p)$ to $M_\lambda(q)$ for the dynamical system defined by (12) for some value of λ between -1 and 1.

Thus, non zero entries in the singular transition matrix can be used to conclude the existence of connecting orbits. A generalization of this argument leads to the following theorem.

Theorem 4.14 *Let $T \in \mathcal{T}_{-1,1}^{\mathrm{sing}}$ and assume $T(q^-, p^+) \neq 0$. Then there exists a finite sequence $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_k > 0$ and $r_{k+1} = p >_{\lambda_k} r_k > \cdots >_{\lambda_2} r_2 >_{\lambda_1} r_1 = q$, elements of $(\mathcal{P}, >)$ where $>_{\lambda_i}$ is the flow defined ordering for the Morse decomposition of (12) at parameter value λ_i.*

This form of transition matrix is by far the most general. In fact, the only necessary assumption is the existence of Morse decompositions of isolated invariant sets which are related by continuation. There is, however, a price to be paid for this generality. Observe that the algebraic properties of T_ϵ for ϵ small are a complete mystery. For example, it is reasonable to ask what is the relation between T_ϵ and $T_{-\epsilon}^{-1}$. Unfortunately the spaces on which T_ϵ and $T_{-\epsilon}$ are defined are different, and hence, even formalizing the question poses some difficulty.

These algebraic questions led to the development of another transition matrix called a *topological transition matrix* denoted by

$$T_{-1,1}^{\mathrm{top}} : \bigoplus_{p \in \mathcal{P}} CH_*(M_1(p)) \to \bigoplus_{p \in \mathcal{P}} CH_*(M_{-1}(p)).$$

Using the same setting as in the discussion of the singular transition matrix, the idea behind topological transition matrices can be described as follows. Since each Morse set $M(p)$ continues over $\mathbf{R}$, there exist isomorphisms

$$F_{-1,1}(p) : CH_*(M_1(p)) \to CH_*(M_{-1}(p)).$$

Similarly, since S continues over $\mathbf{R}$, there is an isomorphism

$$F_{-1,1} : CH_*(S_1) \to CH_*(S_{-1}).$$

Furthermore, if $S_{\pm 1} = \bigcup_{p \in P} M_{\pm 1}(p)$, i.e. if the set of connecting orbits is empty, then there exists an isomorphism

$$\Phi_{\pm 1} : \bigoplus_{p \in P} CH_*(M_{\pm 1}(p)) \to CH_*(S_{\pm 1}).$$

Thus, if there are no connections at $\lambda = \pm 1$, then the following diagram can be constructed:

$$\begin{tikzcd}
\bigoplus_{p \in P} CH_*(M_1(p)) \arrow[r, "\bigoplus_{p \in P} F_{-1,1}(p)"] \arrow[d, "\Phi_1"'] & \bigoplus_{p \in P} CH_*(M_{-1}(p)) \arrow[d, "\Phi_{-1}"] \\
CH_*(S_1) \arrow[r, "F_{-1,1}"] & CH_*(S_{-1})
\end{tikzcd}$$

In general, this diagram is *not* commutative and it is precisely its failure to commute that gives information about connecting orbits. For the purpose of applications it is useful to be able to express this last statement in the form of a matrix. Of course this requires choices of basis on each of the spaces. This can be done as follows. Let $\mathcal{B}_1$ be a basis for $\bigoplus_{p\in P} CH_*(M_1(p))$ and let

$$\mathcal{B}_{-1} = \bigoplus_{p\in P} F_{-1,1}(p)(\mathcal{B}_1)$$

be a basis for $\bigoplus_{p\in P} CH_*(M_{-1}(p))$. Using these bases define the topological transition matrix by

$$T^{\mathrm{top}}_{-1,1} = \Phi_{-1} \circ \bigoplus_{p\in P} F_{-1,1}(p) \circ \Phi_1^{-1}.$$

Observe that this construction only makes sense when the Morse decomposition continues over the entire parameter space, and when there are no connecting orbits with respect to the Morse decomposition at the parameter values $\lambda = \pm 1$.

In this setting it is perfectly clear what is meant by the composition of two transition matrices. It is equally obvious that

$$T^{\mathrm{top}}_{-1,1} = \left(T^{\mathrm{top}}_{1,-1}\right)^{-1}.$$

Furthermore, $T^{\mathrm{top}}_{-1,1}$ shares many properties with elements of $\mathcal{T}^{\mathrm{sing}}_{-1,1}$. In particular, non–zero off diagonal entries imply the existence of connecting orbits for some parameter values between -1 and 1.

An important fact is that when both singular and topological transition matrices are defined then they are "equal". Equal needs to be put in quotation marks since the maps are defined on different vector spaces. However, as is shown in [46] there exists a canonical isomorphism between these spaces which justifies this claim.

Example 4.15 To see how the transition matrices can be used in practice, consider Example 2.8. Again, we use $\mathbf{Z}_2$ coefficients in the calculations. From Example 2.10 we know that the Morse decomposition continues over the interval $c > 0$. Furthermore, Δ_0 and Δ_∞ were calculated in Example 4.12. Thus, augmenting (2) by the equation $\dot{c} = \epsilon(c - c_0)(c - c_\infty)$ where $0 < c_0 << 1$ and $1 << c_\infty$ and applying (14) one obtains

$$\Delta_\epsilon = \begin{bmatrix} 0 & 0 & 0 & T(1^-,1^+) & T(1^-,2^+) & T(1^-,3^+) \\ 0 & 0 & 1 & T(2^-,1^+) & T(2^-,2^+) & T(2^-,3^+) \\ 0 & 0 & 0 & T(3^-,1^+) & T(3^-,2^+) & T(3^-,3^+) \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since, for all $c > 0$ the Lyapunov function implies that $3 > 2 > 1$, and using the fact that Δ_ϵ must be a degree -1 map, this reduces to:

$$\Delta_\epsilon = \begin{bmatrix} 0 & 0 & 0 & T(1^-,1^+) & T(1^-,2^+) & 0 \\ 0 & 0 & 1 & 0 & T(2^-,2^+) & 0 \\ 0 & 0 & 0 & 0 & 0 & T(3^-,3^+) \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the simple form of the flow it is easy to check that $(M(p^-), M(p^+))$ forms an attractor–repeller pair. Furthermore, $CH_*(M(p^-p^+)) = 0$ which implies that $T(p^-,p^+) = 1$. Thus

$$\Delta_\epsilon = \begin{bmatrix} 0 & 0 & 0 & 1 & T(1^-,2^+) & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

However, $\Delta_\epsilon \circ \Delta_\epsilon = 0$, which implies that $T(1^-,2^+) = 1$. Therefore, there exists $c^* > 0$ for which the equation (2) possesses a heteroclinic orbit from $(-1,0)$ to $(1,0)$.

4.1.5 Example - Infinitely Many Connecting Orbits

Consider the system of differential equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \lambda y - x(x-a) \quad x, y, \lambda, a \in \mathbf{R} \\ \dot{\lambda} &= \epsilon(1-\lambda^2) \end{aligned} \tag{15}$$

Observe that for $\epsilon \neq 0$ there are exactly four equilibria $B^\pm := (0,0,\pm 1)$ and $A^\pm := (a,0,\pm 1)$. Our goal is to show that for $\epsilon > 0$ there exist infinitely many connecting orbits from B^- to B^+.

(15) has the same form as (13), suggesting that the singular transition matrix can be used. Also, notice that at $\lambda = \pm 1$ this system has the same form as that of Example 2.3. We begin by studying the case $\epsilon < 0$. The same arguments used in Examples 4.7 and 4.15 give rise to the following result.

Proposition 4.16 *For $\epsilon < 0$, $\mathcal{M}(S_\epsilon) = \{B^\pm, A^\pm\}$ is a Morse decomposition of S_ϵ with an admissible ordering $A^+ > B^+ > B^- > A^-$. Furthermore, using $\mathbf{Z}_2$ coefficients the corresponding connection matrix is*

$$\Delta_\epsilon = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proposition 4.16 implies that the transition matrix is

$$T^{\rm sing}_{-1,1} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right] : CH_*(B^+) \oplus CH_*(A^+) \to CH_*(A^-) \oplus CH_*(B^-).$$

Since Δ_ϵ is a degree -1 map, the only possible non-zero entry is $T^{\rm sing}_{-1,1}(B^-, B^+)$. However, it is easy to check that the segment $B \times (-1,1)$ is an isolated connecting orbit from B^+ to B^-, i.e. it is an element of a separation of $C(B^+, B^-)$. Furthermore, the associated boundary operator is $\partial(B^-, B^+) = 1$. Thus, if this were the only connecting orbit, then $T^{\rm sing}_{-1,1}(B^-, B^+) = 1$. By Theorem 4.10 there must be another connecting orbit in the separation which has a nontrivial boundary operator. Some simple analysis shows such an orbit must wind around the line segment $A \times [-1,1]$.

To simplify the rest of the argument we shall assume that $C(B^+, B^-)$ consists of exactly two orbits; the trivial orbit, $B \times (-1,1)$, and one which winds around the line segment $A \times [-1,1]$ exactly once. For this particular system one can prove this fact using Melnikov methods. In a more general setting one can continue the topological argument presented here allowing for multiple orbits with multiple windings [44].

Returning to the proof of infinitely many connecting orbits, let N be an isolating neighborhood for S_ϵ.

To be precise about what is meant by the winding of orbits consider the following set

$$D = N \backslash (\{A\} \times \mathbf{R}).$$

D is homotopic to S^1, and hence $\pi_1(D) \approx \pi_1(S^1) \approx \mathbf{Z}$ where π_1 denotes the fundamental group. This suggests the following definition.

Definition 4.17 Let γ_ϵ denote a connecting orbit from B^+_ϵ to B^-_ϵ. Then $\gamma_\epsilon \cup (B_\epsilon \times [-1,1])$ generates an element of $\pi_1(D)$ which defines the *winding number* of γ_ϵ.

Let $\widetilde{D}$ denote the universal covering space of D with covering map

$$\rho : \widetilde{D} \to D\,.$$

There exists a unique local flow Φ_ϵ on $\widetilde{D}$ which projects onto φ_ϵ.

Let $\{B^\pm_n \mid n \in \mathbf{Z}\} = \rho^{-1}(B^\pm)$ and let $\widetilde{N} = \rho^{-1}(N)$. Unfortunately, $\widetilde{N}$ is not an isolating neighborhood for Φ_ϵ, since it is not compact. However, one can show that

$$\widetilde{S}_\epsilon(K) := \bigcup_{k=1}^{K} B^\pm_k \;\cup\; \bigcup_{j,k=1}^{K} C(B^+_k, B^-_j; \Phi_\epsilon)$$

is an isolated invariant set.

Using the fact that all the index information of Φ_ϵ must project onto φ_ϵ one can compute the singular transition matrix $\widetilde{T}^{\rm sing}_{-1,1}$ associated with $\widetilde{S}_\epsilon(K)$.

Since we are assuming that there is the trivial connection and exactly one other connection with winding number 1 from B^+ to B^- this implies that

$$\widetilde{T}^{\text{sing}}_{-1,1} = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & & 0 \\ & & \ddots & \ddots & & \\ & & & & & \\ & & & 0 & 1 & 1 \\ 0 & & \dots & & 0 & 1 \end{bmatrix}.$$

Observe that there are no connecting orbits at $\lambda = \pm 1$. Therefore, we can apply the topological transition matrix. As was indicated in the previous section, when both topological and singular transition matrices are defined on the same Morse decomposition they agree. Hence, modulo some canonical isomorphisms one may write

$$\begin{aligned} \widetilde{T}^{\text{sing}}_{1,-1} = \widetilde{T}^{\text{top}}_{1,-1} = \left(\widetilde{T}^{\text{top}}_{-1,1}\right)^{-1} &= \left(\widetilde{T}^{\text{sing}}_{-1,1}\right)^{-1} \\ &= \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 1 & \dots & & 1 \\ & & \ddots & \ddots & & \\ & & & & & \\ & & & 0 & 1 & 1 \\ 0 & & \dots & & 0 & 1 \end{bmatrix}. \end{aligned}$$

But this implies that for $\epsilon > 0$, there exist the desired connecting orbits with arbitrary winding numbers. Under ρ each orbit with a different winding number projects to a different solution of (15). Therefore, there are an infinite number of distinct connecting orbits.

4.2 Periodic Orbits

Given an isolated invariant set S with a Morse decomposition $\mathcal{M}(S)$, Theorem 2.11 implies that the only type of dynamical objects that lie in S, but not within the Morse sets are connecting orbits. The theme of Section 4.1 was how to use the Conley index to prove the existence of and describe the structure of these solutions. Theorem 2.19 implies that all dynamics in an invariant set either consist of connecting orbits or lie in the chain recurrent set. With this in mind we now turn to the question of how can the Conley index be used to understand recurrent dynamics. The simplest form of recurrence is that of a periodic orbit. As Example 3.18 demonstrates it is not enough to know that the index of an invariant set S is that of a periodic orbit to conclude that S contains a periodic orbit. What is surprising, however, is that the only additional necessary hypothesis is that the invariant set have a minimal amount of recurrence.

4.2.1 Poincaré Section

To insure recurrence we shall make use of a Poincaré section. Since we are adopting the point of view that it is the isolating neighborhood which is the observable object rather than the invariant set, we want to formulate all of our hypotheses in terms of N, rather than S. Therefore, we employ the following definition.

Definition 4.18 $\Xi \subset X$ is a *Poincaré section* for N under φ if Ξ is a local section, $\Xi_N := \Xi \cap N$ is closed, and for every $x \in N$ there exists $t_x > 0$ such that

$$\varphi(t_x, x) \in \Xi. \tag{16}$$

Observe that it is not necessary to know S in order to find a Poincaré section. Also, Ξ is not required to be a subset of N. Indeed, if N has an exit set, no subset of N can be a Poincaré section, as there will be points in N whose orbits exit N before they cross the section.

The following example is included to indicate that this definition is sufficiently weak to make it possible to prove the existence of Poincaré section in the context of a differential equation.

Example 4.19 The Nagumo equations (see [76] for a derivation of the equations) are given by,

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= \theta v - f(u) + w \\ \dot{w} &= \frac{\epsilon}{\theta} u,\end{aligned} \tag{17}$$

The nonlinearity f is a cubic like function as indicated in Figure 3. In this problem, θ is treated as a constant rather than as a parameter value. φ_ϵ denotes the flow on $\mathbf{R}^3$ generated by (17) for a fixed ϵ.

For $\epsilon = 0$ the Nagumo equation becomes a parameterized family of differential equations in the plane with parameter w. In particular, for each fixed w one has a flow

$$\psi_w : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$$

generated by the *fast system*

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= \theta v - f(u) + w.\end{aligned} \tag{18}$$

At $\epsilon = 0$ the equilibria of (17) are given by the set $\{(u, 0, w) \mid f(u) = w\}$. Let $[0, w_1]$ be an interval such that for $c \in [0, w_1]$, there are three branches of solutions to $f(u) = c$ (see Figure 3). Observe that for each fixed value, $w = c$, the dynamics of (18) is similar to that described in Example 2.8. In particular, the left and right roots of $f(u) = c$ give rise to saddle points in flow ψ_c. Extending this argument to all $w \in [w_0, w_1]$ one obtains two parameterized

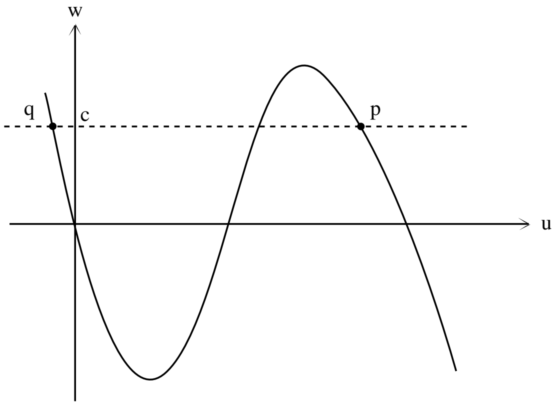

Figure 3: The nonlinearity of the Nagumo equation.

branches of equilibria $q(w)$ and $p(w)$, respectively. For an appropriate choice of θ one can prove, using the same type of argument employed in Example 4.15, that there are parameter values w_* and w^*, with $0 < w_* < w^* < w_1$, for which there are heteroclinic connections γ_* from $q(w_*)$ to $p(w_*)$ and γ^* from $p(w^*)$ to $q(w^*)$, respectively.

Let Q be the set of equilibria given by $q(w)$ for $w \in [w_0, w_1]$ and P be the set of equilibria given by $p(w)$ for $w \in [w_0, w_1]$. Now observe that for $\epsilon > 0$, $\dot{w} < 0$ along the curve Q and $\dot{w} > 0$ along the curve P.

This suggests that for each fixed $0 < \epsilon << 1$ there may exist a periodic orbit of φ_ϵ close to the segments $\gamma_* \cup P \cup \gamma^* \cup Q$. To prove this using the index theory requires several steps. The first is to find an appropriate isolating neighborhood by choosing tubular neighborhoods around each of the segments. More precisely, let T_P and T_Q be tubular neighborhoods around the segments P and Q, and let T_* and T^* be tubular neighborhoods around the segments γ_* and γ^*. Set $N = T_P \cup T_Q \cup T_* \cup T^*$. It can be shown using the techniques described in Section 4.4 that N is an isolating neighborhood for $0 < \epsilon << 1$. The second step is to compute the index of $S_\epsilon = \mathrm{Inv}(N, \varphi_\epsilon)$ wherein one obtains

$$CH_k(S_\epsilon) \approx \begin{cases} \mathbf{Z}_2 & \text{if } k = 1, 2; \\ 0 & \text{otherwise.} \end{cases}$$

The techniques for doing this are also discussed in Section 4.4. The final step is to apply Theorem 4.20 and conclude the existence of a periodic orbit. To do this it must be shown that N has a Poincaré section which can be done as follows.

Since N is an isolating neighborhood, by Theorem 3.6 there is an isolating block B that lies within an arbitrarily small neighborhood of S_ϵ such $S_\epsilon = \mathrm{Inv}(B, \varphi_\epsilon)$. Furthermore, it can be shown (see [75]) that for any time $T > 0$ there is an isolating block E such that:

1. $\mathrm{Inv}(E, \varphi_\epsilon) = \mathrm{Inv}(B, \varphi_\epsilon)$,
2. $E \subset \mathrm{int}(B)$, and
3. $x \in E$ implies $\varphi_\epsilon([0,T], x) \subset B$.

Observe that it is sufficient to show that E has a Poincaré section and then apply Theorem 4.20 to $\mathrm{Inv}\, E$.

Let $K = \{(u,v,w) \in B \cap T_Q \mid (w^* + w_*)/2 - \delta < w < (w^* + w_*)/2 + \delta\}$ for $\delta > 0$, but small. and let $K_0 = \{(u,v,w) \in B \cap T_Q \mid w = (w^* + w_*)/2\}$. Since $\dot{w} < 0$ along the curve Q, choosing B within a sufficiently small neighborhood of S_ϵ implies that K_0 is a local section.

Clearly if (u,v,w) stays in E for all forward time under φ^ϵ, then its forward orbit passes through the tubular neighborhoods in the periodic progression of $T_Q \to T_* \to T_P \to T^* \to T_Q$. This is because the flow in each tubular neighborhood is determined by the flow near the segments. Choose the time T mentioned about to be longer than the time needed for any orbit in S_ϵ to pass through all four tubular neighborhoods.

Observe that if $S_\epsilon = \emptyset$, then B^- is a Poincaré section for E. Thus, what needs to be dealt with are those points that remain in E for all positive time.

Let

$$\Xi := (B^- \setminus K) \cup K_0.$$

We claim that Ξ is a Poincaré section for E. $E \cup \Xi$ is clearly closed. It is a section to the flow because both B^- and K_0 are. Finally, we must show that the forward orbit of every point in E intersect Ξ. Clearly if (u,v,w) stays in E for all forward time under φ^ϵ its forward orbit intersects Π. Thus, we only need to worry about points (u,v,w) for which there exists a time $\tau_0 > 0$ such that $\varphi_\epsilon(\tau_0, (u,v,w)) \notin E$. Since $\mathrm{Inv}(E, \varphi_\epsilon) = \mathrm{Inv}(B, \varphi_\epsilon)$, this is equivalent to the existence of $\tau_1 > 0$ such that $\varphi_\epsilon(\tau_1, (u,v,w)) \notin B$. This in turn implies that there exists $\tau_2 > 0$ such that $\varphi_\epsilon(\tau_1, (u,v,w)) \in B^-$. If $\varphi_\epsilon(\tau_1, (u,v,w)) \in B^- \setminus K$ then $\varphi_\epsilon(\tau_1, (u,v,w)) \in \Xi$. So we have further reduced the problem to $\varphi_\epsilon(\tau_1, (u,v,w)) \in B^- \cap K$. Observe, however, that $(u,v,w) \in E$ implies that there exists a positive time $0 < s < T$ such that $\varphi_\epsilon([0,s], (u,v,w)) \subset B$ and $\varphi_\epsilon(s, (u,v,w)) \in K_0$.

4.2.2 Existence of Periodic Orbits

The following theorem provides very general conditions under which an isolated invariant set must contain a periodic orbit.

Theorem 4.20 *If N is an isolating neighborhood which admits a Poincaré section Ξ and for all $n \in \mathbf{Z}$ either*

$$\dim\, CH_{2n}(\mathrm{Inv}\, N) = \dim\, CH_{2n+1}(\mathrm{Inv}\, N) \tag{19}$$

or

$$\dim\, CH_{2n}(\mathrm{Inv}\, N) = \dim\, CH_{2n-1}(\mathrm{Inv}\, N) \tag{20}$$

where not all the above dimensions are zero, then $\operatorname{Inv} N$ *contains a periodic trajectory.*

Corollary 4.21 *Under the above hypothesis, if* $\operatorname{Inv} N$ *has the Conley index of a hyperbolic periodic orbit, then* $\operatorname{Inv} N$ *contains a periodic orbit.*

A key step in the proof of Theorem 4.20 is the following exact sequence which relates the index of an isolated invariant set which admits a Poincaré section with the index of the invariant set under the Poincaré map.

Theorem 4.22 *Assume N is an isolating neighborhood for the flow φ and assume that N admits a Poincaré section Ξ. Let Π denote the corresponding Poincaré map, $S = \operatorname{Inv}(N, \varphi)$, and $K = \Xi \cap S$. Then, there is the following exact sequence of homology Conley indices:*

$$\cdots \to CH_n(S,\varphi) \to CH_n(K,\Pi) \stackrel{id-\chi_n(K,\Pi)}{\longrightarrow} CH_n(K,\Pi) \to CH_{n-1}(S,\varphi) \to \cdots \tag{21}$$

As was indicated in the introduction and as the following simple examples will demonstrate the Poincaré map may carry more information than the index of the flow.

Example 4.23 Let N be an isolating neighborhood for a flow φ and let $S = \operatorname{Inv}(N, \varphi)$. Assume that N admits a Poincaré section Ξ. Let Π be the induced Poincaré map. Furthermore, assume that Π is the G-horseshoe map of Example 3.36. Let $K = \operatorname{Inv}(\Xi, \Pi)$. If we now apply Theorem 4.22, then the only nontrivial portion of (21) is

$$\cdots \to CH_1(S,\varphi) \to CH_1(K,\Pi) \stackrel{id-\chi_1(K,\Pi)}{\longrightarrow} CH_1(K,\Pi) \to CH_0(S,\varphi) \to \cdots \tag{22}$$

Observe that $id - \chi_1(K,\Pi) = 1 - 2 = -1$, and hence (22) becomes

$$\cdots \to 0 \to CH_1(S,\varphi) \to Q \stackrel{-1}{\longrightarrow} Q \to CH_0(S,\varphi) \to 0 \to \cdots$$

Exactness of this sequence implies that $CH_*(S,\varphi) = 0$.

It is worth emphasizing just how much information has been lost in the flow index. Since $CH_*(S,\varphi) = 0$, we can draw no conclusion from this information. Since the map index is nontrivial, the Ważewski principle implies that K and hence S is nonempty. In fact, however, the map index can be used to say much more about the structure of K. In particular, the fact that $\chi_1(K,\Pi) = 2$ allows one, by the direct application of a result of A. Baker [1], to conclude that the entropy of K is bounded from below by $\ln 2$.

On a more general note, as will be made clear in Section 4.3, the spectrum of $\chi_*(K,\Pi)$ can be used to conclude the existence of rather complicated orbit structure in K. From (21) it is clear that any information carried by spectral values different from 1 is lost in the flow defined index.

4.3 The topology of S

As was discussed in the introduction, the topology of invariant sets can change dramatically under even small perturbations. Since the Conley index is stable with respect to perturbations one cannot expect that knowledge of the index will immediately give information about the structure of the invariant set. In fact, even if the Conley index of S is extremely complicated, it is possible that S consists of a single equilibrium point. In Section 4.1, we were able to obtain additional information about the structure of S by assuming knowledge of an index filtration. In Section 4.2, the existence of periodic orbits could be concluded if one knew that there was a Poincaré section for the isolating neighborhood. Another approach, which we now describe is due to Floer [18] and is based on the idea of restricting the topology of the isolating neighborhood.

To be more precise, let $\varphi_\lambda : \mathbf{R} \times X \to X$ be a parameterized family of dynamical systems with $\lambda \in [0,1]$. Let S_0 be a normally hyperbolic invariant set for φ_0 and let N be a tubular neighborhood for S_0. Then N is an isolating neighborhood for S_0. There are two observations to make at this point. The first is that any retract $\alpha : N \to S$ is homotopic to the canonical projection of N onto S coming from the tubular neighborhood structure. The second is that the homotopy Conley index of S is essentially the Thom complex for the unstable disk bundle for S. Using this Floer obtains the following theorem.

Theorem 4.24 *Assume that N is an isolating neighborhood for φ_λ for all $\lambda \in [0,1]$. Then*

$$(\alpha\mid_{S_1})^* : H^*(S_0) \to H^*(S_1)$$

is an injection

In other words, the topology of the invariant set can only increase as the flow is continued. The simplest example of this phenomenon is to consider a hyperbolic saddle in a planer flow. This is clearly a hyperbolic invariant set with a disk as a tubular neighborhood. Now consider a continuation of the flow that preserves the hyperbolic equilibrium but introduces a degenerate fixed point on one of the trajectories that passes close to the equilibrium. This new isolated invariant set consists of two points, thus the topology of the isolated invariant set has become more complicated.

This result has far reaching consequences. This lower bound on the cohomology of the invariant set can be used to estimate the cup length of S_1. In turn, if one knows that φ_1 is a gradient flow, then one has a lower bound on the number of critical points of the potential.

It should be noted that in [18] Floer showed that this construction can be carried out in the setting of equivariant dynamics. Work in a similar spirit was also carried out by Poźniak [67].

4.4 Symbolic Dynamics

In Section 3.2 we saw that the Conley index of a U-horseshoe is trivial. This is not surprising, because one may easily continue the horseshoe to the empty set. This also means that, similar to the case of periodic orbits, the index itself is not sufficient to reconstruct the horseshoe dynamics. So far we do not know what assumptions, if any, would guarantee the existence of an embedded horseshoe. However it is possible to formulate assumptions in terms of the Conley index which imply the existence of a semiconjugacy onto shift dynamics, i.e. a continuous map onto bi-infinite sequences of prescribed set of symbols, which commutes with the shift dynamics on the sequences. To be more precise, let $\Sigma_k := \{ a : \mathbf{Z} \to \{1, 2, \ldots k\} \}$ denote the space of bi-infinite sequences on k symbols with the product topology. Let $\sigma : \Sigma_k \to \Sigma_k$ be the shift map given by

$$(\sigma(a))_n := a_{n+1}.$$

The semiconjugacy onto the shift dynamics is a continuous surjective map $\rho : \mathrm{Inv}(N, f) \to \Sigma_k$ such that the diagram

$$\begin{tikzcd} \mathrm{Inv}(N,f) \arrow[r, "f"] \arrow[d, "\rho"] & \mathrm{Inv}(N,f) \arrow[d, "\rho"] \\ \Sigma_k \arrow[r, "\sigma"] & \Sigma_k \end{tikzcd} \tag{23}$$

commutes.

Returning to Example 3.27 we easily see that replacing the isolating neighborhood N by the set $R := R_0 \cup R_1$ we obtain another isolating neighborhood which isolates the same isolated invariant set. This suggests how the semiconjugacy map may be defined in situations when the isolating neighborhood decomposes into a number of disjoint compact components as in the case of R. Namely, if $N = N_0 \cup N_1 \cup \ldots \cup N_k$ is a union of k disjoint compact sets then we define the map $\tau : N \to \{1, 2, \ldots k\}$ by $\tau(x) = i$ if and only if $x \in N_i$ and the map $\rho : \mathrm{Inv}(N, f) \to \Sigma_k$ by $\rho(x) := \{ \tau(f^i(x)) \}_{i=-\infty}^{\infty}$. The map ρ is continuous and the diagram (23) commutes. The hard part in completing the construction of the semiconjugacy is to show that ρ is surjective. This is guaranteed by the following theorem.

Theorem 4.25 *Let $f : X \to X$ be a homeomorphism on a locally compact metric space. Assume that $N = N_0 \cup N_1$ is an isolating neighborhood under f where N_0 and N_1 are disjoint compact sets and for $k = 0, 1$*

$$\mathrm{Con}_n(N_k) = \begin{cases} (\mathbf{Q}, \mathrm{id}) & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, $N_{kl} := (N_k \cap f(N_k)) \cup (N_k \cap f(N_l)) \cup (N_l \cap f(N_l))$ for $k, l \in \{0, 1\}, k \neq l$, are isolating neighborhoods. If additionally $\chi^*(N_{kl})$ is not conjugate to the identity, then there exists a $d \in \mathbf{N}$ and a continuous surjection $\rho : \mathrm{Inv}(N, f) \to \Sigma_2$ such that the following diagram

$$\begin{tikzcd} \mathrm{Inv}(N,f) \arrow[r, "f^d"] \arrow[d, "\rho"] & \mathrm{Inv}(N,f) \arrow[d, "\rho"] \\ \Sigma_2 \arrow[r, "\sigma"] & \Sigma_2 \end{tikzcd} \tag{24}$$

commutes.

A disadvantage in the above theorem is the fact that one needs four isolating neighborhoods and the corresponding Conley indices to verify the existence of a semiconjugacy onto the shift dynamics. Below we present a theorem which shows that two isolating neighborhoods are sufficient.

Given a linear operator M let $\operatorname{spec}(M)$ denote the set of eigenvalues of M where the eigenvalues are repeated according to their multiplicity. Given two sets A and B, their *amalgamation* $A \amalg B$ is obtained by taking the union of A and B but treating elements common to both A and B as distinct elements in $A \amalg B$. For example $\{1,2,2,3\} \amalg \{0,0,2,3\} = \{0,0,1,2,2,2,3,3\}$. A set $Q \subset \mathbf{C}$ is *cyclic* if $Q = \sum_{i=1}^{p} Q_i$ where $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and

$$Q_i = \{ z \in \mathbf{C} \mid z^{n_i} = m_i \text{ for some } n_i \in \mathbf{Z}^+, m_i \in \mathbf{Z} \}.$$

Theorem 4.26 *Let $f : X \to X$ be a homeomorphism on a locally compact metric space. Assume that $N \subset X$ is an isolating neighborhood which is the disjoint union of compact sets N_0 and N_1, i.e.*

$$N = N_0 \cup N_1.$$

Let $S := \operatorname{Inv}(N, f)$ and $S_i := \operatorname{Inv}(N_i, f)$, $i = 0, 1$. If for some positive integer n

$$\operatorname{spec}(\chi_n(S_0)) \amalg \operatorname{spec}(\chi_n(S_1)) \not\subset \operatorname{spec}(\chi_n(S))$$

or

$$\operatorname{spec}(\chi_n(S_0)) \backslash (\operatorname{spec}(\chi_n(S_0)) \amalg \operatorname{spec}(\chi_n(S_1)))$$

is not acyclic, then there exists a positive integer d and a continuous surjection $\rho : S \to \Sigma_2$ such that diagram (24) commutes.

Recently A. Szymczak [79] presented a generalization of the Conley index theory which keeps track of the components of the isolating neighborhood. The generalization uses index pairs satisfying certain extra conditions and allows one to prove a criterion for a semiconjugacy onto the shift dynamics on k symbols restricted to a certain transition matrix which may be deduced from the generalized Conley index. The theorem requires the index computation for only one isolating neighborhood and additionally, guarantees the existence of periodic points corresponding to every periodic sequence of symbols. Unfortunately, the presentation of this generalization of the Conley index is beyond the scope of this paper.

4.5 Fast Slow Systems

We finish this section with a discussion of a new direction of development in the Conley index theory. In Example 4.19 a Poincaré section for an isolating neighborhood was described. Two points were asserted. The first was that the region described is an isolating neighborhood, and the second was that the index of the isolated invariant set is known. We shall now briefly mention the techniques used to obtain these results. The reader is referred to [8, 58, 32] for further details.

The general framework is that of a family of differential equations on $\mathbf{R}^n$ given by

$$\dot{x} = f(x,\epsilon) = f_0(x) + \sum_{i=1}^{m} \epsilon^i f_i(x) + o(\epsilon^m)$$

where $\epsilon \geq 0$. For each ϵ let $\varphi_\epsilon : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ denote the flow generated by (1) with parameter value ϵ. The reader can check that Examples 4.19 and 4.1.5 are of this form. In both cases for $\epsilon = 0$ there is no isolating neighborhood for the dynamics that is of interest for $\epsilon > 0$. These suggests the following definition.

Definition 4.27 A compact set $N \subset \mathbf{R}^n$ is called a *singular isolating neighborhood* if N is not an isolating neighborhood for φ_0, but there is an $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon}]$, N is an isolating neighborhood for φ_ϵ.

The first question that should be addressed is how can one verify that a compact set is a singular isolating neighborhood. The answer requires the following definitions.

Definition 4.28 The *average of g on S*, $\mathrm{Ave}(g,S)$, is the limit as $T \to \infty$ of the set of numbers $\{\frac{1}{T}\int_0^T g(\varphi_0(s,x))\, ds \mid x \in S\}$. g has *strictly positive averages* on S if $\mathrm{Ave}(g,S) \subset (0,\infty)$.

Definition 4.29 A point $x \in S$ is a *C-slow exit point* if there exists a compact set $K_x \subset S$ invariant under φ_0, a neighborhood U_x of $\mathcal{R}(K_x)$, an $\bar{\epsilon} > 0$ and a function $\ell : \mathrm{cl}(U_x) \times [0,\bar{\epsilon}] \to \mathbf{R}$ such that the following conditions are satisfied.

1. $\omega(x,\varphi_0) \subset K_x$.
2. ℓ is of the form $\ell(z,\epsilon) = \ell_0(z) + \epsilon\ell_1(z) + \ldots + \epsilon^m \ell_m(z)$.
3. If $L_0 = \{\, z \mid \ell_0(z) = 0 \,\}$ then $K_x \cap \mathrm{cl}(U_x) = S \cap L_0 \cap \mathrm{cl}(U_x)$ and furthermore $\ell_0|_{S\cap cl(U_x)} \leq 0$.
4. Let $g_j(z) = \nabla_z \ell_0(z) \cdot f_j(z) + \nabla_z \ell_1(z) \cdot f_{j-1}(z) + \ldots + \nabla_z \ell_j(z) \cdot f_0(z)$. Then for some m, $g_j \equiv 0$ if $j < m$, and g_m has strictly positive averages on $\mathcal{R}(K_x)$.

A *C-slow entrance point* is a point which is a C-slow exit point in the backward flow.

Given a compact set $N \subset \mathbf{R}^n$ let $S = \mathrm{Inv}(N, \varphi_0)$. Let S^- (S^+) denote the set of C-slow exit (entrance) points. Set $S_\partial = S \cap \partial N$ and $S_\partial^\pm = S_\partial \cap S^\pm$.

Theorem 4.30 *If $S \cap \partial N \subset S^\pm$, then N is a singular isolating neighborhood.*

Following the standard assumption underlying the technique of singular perturbations, it is assumed that the dynamics of the "singular" system $\dot{x} = f_0(x)$ is simpler to analyze than that of the perturbed system. Thus, the goal is to determine the Conley indices for isolated invariant sets for small $\epsilon > 0$ from the dynamics of φ_0. This motivates the following definition.

Definition 4.31 A pair of compact sets (N, L) with $L \subset N$ is a *singular index pair* if $\mathrm{cl}(N \setminus L)$ is a singular isolating neighborhood, and there exists $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon}]$

$$H_*(N, L) \cong CH_*(\mathrm{Inv}(\mathrm{cl}(N \setminus L), \varphi_\epsilon)).$$

Again the crucial point is to obtain results which verify that a given pair is a singular index pair. The simplest approach is to mimic the three conditions in the definition of an index pair as closely as possible. With regard to the exit set requirement, it is clear that L will need to contain

$$N^- := \{x \in \partial N \mid \varphi_0((0,t), x) \not\subset N \text{ for all } t > 0\}.$$

Positive invariance leads to the following construction. Given $Y \subset N$ its *push forward set* in N under the flow φ_0 is defined to be

$$\rho(Y, N) := \{x \in N \mid \exists z \in Y,\, t \geq 0 \text{ such that } \varphi_0([0,t], z) \subset N,\ \varphi_0(t, z) = x\}.$$

Observe that $\rho(Y, N)$ consists of the set of points which can be reached from Y by a positive trajectory in N and that $Y \subset \rho(Y, N)$. The *unstable set* of an invariant set $Y \subset N$ under φ_0 is

$$W_N^u(Y) := \{x \in N \mid \varphi_0((-\infty, 0), x) \subset N \text{ and } \alpha(x, \varphi_0) \subset Y\}.$$

Notice that $Y \subset W_N^u(Y)$.

It is easy to find examples of isolating neighborhoods N for which there does not exist a set L such that (N, L) is an index pair for $\mathrm{Inv}\, N$. The following condition prevents this.

A slow entrance point x is a *strict slow entrance point* if there exists a neighborhood Θ_x of x and an $\bar{\epsilon} > 0$ such that if $y \in \Theta_x \cap N$ and $\epsilon \in (0, \bar{\epsilon}]$, then there exists $t_y(\epsilon) > 0$ for which $\varphi_\epsilon([0, t_y(\epsilon)], y) \subset N$. Let S_∂^{++} denote the strict slow entrance points.

Theorem 4.32 *Let N be a singular isolating neighborhood. Assume*

1. *S_∂^- consists of C-slow exit points.*
2. *$S_\partial \subset S_\partial^{++} \cup S_\partial^-$.*
3. *$(S_\partial^{++} \setminus S_\partial^-) \cap \mathrm{cl}(N^-) = \emptyset$.*

For each $x \in S_\partial^-$, let K_x denote a compact invariant set as in Definition 4.29. Define

$$L := \rho(cl(N^-), N, \varphi_0) \bigcup W_N^u \left(\bigcup_{x \in S_\partial^-} \mathcal{R}(K_x) \right).$$

If L is closed, then (N, L) is a singular index pair for the family of flows φ_ϵ.

Notice that all the computations used in the construction of L involve only the singular flow. The higher order terms in (1) are only used in determining which points are slow exit and slow entrance points.

4.6 References

One of the pioneering applications of the Conley index is due to Conley and J. Smoller and involved the proof of the existence of shocks waves [11, 12, 13, 14]. The proofs were based on arguments involving attractor-repeller pairs. For a more complete discussion on this topic see [76]. Since then there have been a large number of applications of these techniques to finding traveling waves [10, 30, 51, 53, 60]

The idea of the connection matrix was due to Conley, but its existence was first established by R. Franzosa [25, 26, 27, 28]. It has since been generalized to the setting of maps, first by Robbin and Salamon [71] and later by Richeson [69].

As was mentioned in the text transition matrices were first developed in the singular form [68, 50]. The topological transition matrices were defined in [46]. A purely algebraic development of the transition matrix can be found in [29].

The results on the existence of periodic orbits and the long exact sequence relating the index of the Poincaré map with that of the flow can be found in [48]

The first characterization of symbolic dynamics in terms of the Conley index was given in [55]. Theorem 4.26 based on the spectrum of the Conley index was proved in [5]. The optimality of this result is shown in [6].

5 Multivalued dynamics.

5.1 Multivalued maps.

The theorems in Section 4.4 characterizing chaos in terms of the Conley index cannot be applied directly to differential equations, because the automorphism part of the index is always the identity in this case. However, there is hope that they could be applied to a Poincaré map if such a map exists. The problem is that in most concrete applications our knowledge of the Poincaré map is based mostly on numerical experiments.

The only way to obtain rigorous information from numerical computations is to perform rigorous error analysis by means of interval arithmetic. As an outcome one obtains a multivalued map: the exact values are not known but sets enclosing the exact values are computed rigorously. It turns out that the multivalued map may carry sufficient information to make rigorous claims about the dynamics of the differential equation.

To keep things simple we will study only dynamics in compact subspaces of Euclidean space. Let $X, Y \subset \mathbf{R}^d$ be two such spaces. Formally, a *multivalued (mv) map* from X to Y is a map $F : X \to \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ stands for the family of all subsets of Y. To emphasize that we think of F as a generalization of a single valued map, we will write $F : X \rightrightarrows Y$ to denote a mv map from X to Y. We will also consider mv maps of pairs. We say that $F : (X, X_0) \rightrightarrows (Y, Y_0)$ maps (X, X_0) into (Y, Y_0) if $F : X \rightrightarrows Y$ is a mv map such that $F(X_0) \subset Y_0$. We assume that all mv maps considered in this paper have non-empty compact values.

For $A \subset X$ we define the *image* of A under F by

$$F(A) := \bigcup \{ F(x) \mid x \in A \}.$$

For $B \subset Y$ we define the *weak preimage* of B under F by

$$F^{*-1}(B) := \{ x \in X \mid F(x) \cap B \neq \emptyset \}.$$

and the strong preimage by

$$F^{-1}(B) := \{ x \in X \mid F(x) \subset B \}.$$

The *inverse* of a multivalued map $F : X \rightrightarrows Y$ is a mv map $F^{-1} : Y \rightrightarrows X$ defined by $F^{-1}(y) := \{ x \in X \mid y \in F(x) \}$.

We say that F is *upper semicontinuous (usc)* if $F^{-1}(U)$ is open for any open $U \subset Y$. By a *selector* of F we mean a single-valued continuous map $f : X \to Y$ such that $f(x) \in F(x)$ for every $x \in X$. Sometimes we will also consider mv selectors. A *mv selector* is a mv map $G : X \rightrightarrows Y$ such that $G(x) \subset F(x)$ for every $x \in X$.

5.2 Representable multivalued maps.

The computer can handle only a finite amount of information. In particular all arithmetic operations in the computer are performed on a finite subset $\hat{\mathbf{R}} \subset \bar{\mathbf{R}} := \mathbf{R} \cup \{ -\infty, \infty \}$. We will call $\hat{\mathbf{R}}$ a *representation* of $\bar{\mathbf{R}}$ and the elements of $\hat{\mathbf{R}}$ *representable numbers*. In concrete realizations of computer hardware the set $\hat{\mathbf{R}}$ usually contains $-\infty, 0, \infty$ and a certain set of binary fractions. Since no non-trivial finite set is closed under arithmetic operations, rounding is used whenever the result of an operation is not in $\hat{\mathbf{R}}$. To keep rigor in computations, instead of performing the usual rounding to the nearest representable number, the result may be stored as a pair of two representable numbers: a lower and an upper bound obtained via rounding down and rounding up respectively. The process is known as *interval computations* or *interval arithmetic* and may be easily implemented on most computers. We say that a representation $\hat{\mathbf{R}}_1 \subset \bar{\mathbf{R}}$ is a refinement of $\hat{\mathbf{R}}$ if for any $x, y \in \hat{\mathbf{R}}$ there exists a $z \in \hat{\mathbf{R}}_1$ such that $x < z < y$.

To work with multivalued maps on the computer we need the class of representable sets and representable multivalued maps. An *$\hat{\mathbf{R}}$-representable interval set* in $\mathbf{R}^d$ is a Cartesian product of d intervals in $\mathbf{R}$ with endpoints in $\hat{\mathbf{R}}$. An *elementary $\hat{\mathbf{R}}$-representable set* is a representable interval set which cannot be decomposed as the union of two non-empty representable interval sets. An *$\hat{\mathbf{R}}$-representable set* is a finite union of elementary representable sets.

A nice feature of the class of representable sets is that it is closed under the set-theoretic union, intersection and difference as well as under the operations of taking the interior or closure. This means that any finite combination of these operations on representable sets is algorithmizable.

A mv map $F : X \rightrightarrows Y$ is *$\hat{\mathbf{R}}$-representable* if for every $x \in X$ the set $F(x)$ is $\hat{\mathbf{R}}$-representable and F restricted to any elementary representable set is constant.

The image and preimage of a representable set may be computed algorithmically as the following proposition shows.

Proposition 5.1 *The image and preimage of an $\hat{\mathbf{R}}$-representable set in an $\hat{\mathbf{R}}$-representable mv map is $\hat{\mathbf{R}}$-representable.*

In the sequel we will usually drop the prefix $\hat{\mathbf{R}}$ when speaking about representable sets and numbers, although in applications it is usually necessary to work simultaneously with several representations. When working with representable numbers usually it is sufficient to take as $\hat{\mathbf{R}}$ the representation embedded in hardware, though sometimes it is necessary to implement via suitable software larger sets of representable numbers. On the other hand, when working with representable set and mv maps, one usually takes the set of representable numbers which is much smaller than representable numbers embedded in hardware. The reason is that representable sets and mv maps are stored as lists and then large $\hat{\mathbf{R}}$ results in very large lists.

5.3 Approximation and inheritance.

As we already mentioned, we would like to study the properties of single valued maps by enclosing them in multivalued maps. We say that $F : X \rightrightarrows Y$ is an *$\hat{\mathbf{R}}$-representation of* $f : U \to Y$ if $U \subset X$, F is $\hat{\mathbf{R}}$-representable and f is a *selector* of $F_{|U}$.

Assume that $\{\hat{\mathbf{R}}_n\}_{n\in\mathbf{N}}$ is a given, monotonically increasing sequence of representations of $\mathbf{R}$ such that $\hat{\mathbf{R}}_{n+1}$ is a refinement of $\hat{\mathbf{R}}_n$ and $\bigcup_{n\in\mathbf{N}} \hat{\mathbf{R}}_n$ is dense in $\mathbf{R}$. A natural question is whether given a continuous map $f : U \to Y$ it is possible to find a sequence $\{F_n\}_{n\in\mathbf{N}}$ of $\hat{\mathbf{R}}_n$-representations of f such that $F_n \to f$, i.e. for every $x \in X$ and for every $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that

$$n \geq N, y \in F_n(x) \;\Rightarrow\; \operatorname{dist}(y, f(x)) \leq \epsilon.$$

A positive answer to the above question gives the following simple theorem.

Theorem 5.2 *If $f : U \to Y$ is L-Lipschitz continuous, then there exists a sequence $\{F_n\}_{n\in\mathbf{N}} : X \rightrightarrows Y$ of usc convex-valued representations of f such that $F_n \to f$.*

The above theorem shows that, given a sufficiently powerful computer, any Lipschitz continuous map admits an arbitrarily close multivalued representation. In other words Lipschitz maps may be approximated by representable mv maps. However, this approximation has a feature which distinguishes it from classical approximations. In classical approximations there is no way to extract properties of the approximated function from just one or any finite number of approximating functions. In case of mv approximations it is sometimes possible because of a feature we call inheritance.

To explain this simple idea assume $\mathcal{A}$ is a collection of mv maps and $\varphi(F)$ is a property of such maps. We say that φ is *inheritable* if for every single-valued selector f of F in $\mathcal{A}$

$$\varphi(F) \;\Rightarrow\; \varphi(f)$$

We say that an inheritable property φ is *strongly inheritable* if for any single valued map $f \in \mathcal{A}$ such that $\varphi(f)$ and for any sequence $\{F_n\} \subset \mathcal{A}$ satisfying $F_n \to f$ we have $\varphi(F_n)$ for n sufficiently large.

If $\alpha(F)$ is a term then we say that α is inheritable (strongly inheritable) if for any x the property $\alpha(F) = x$ is inheritable (strongly inheritable).

As an example consider the family $\mathcal{A} = \{F : \mathbf{R} \rightrightarrows \mathbf{R}\}$. Then the property

$$\alpha(F) \;\Leftrightarrow\; \exists x \in \mathbf{R}\; F(x) > 0$$

is strongly inheritable. Replacing the strong inequality in the above example by a weak inequality we obtain a property which is inheritable but is not strongly inheritable. A simple example of a non-inheritable property is

$$\gamma(F) \;\Leftrightarrow\; \exists x \in \mathbf{R}\; F(x) \ni 0.$$

Most properties of interest are non-inheritable. However there are cases when one can convert inheritable properties into non-inheritable properties via suitable theorems. For instance a theorem, which converts properties like α in the above example to property γ, is the Darboux Theorem. Note that in case of single-valued maps property γ reduces to the existence of a zero of a function. Thus representable mv maps together with the Darboux Theorem provide a way of verifying algorithmically the existence of a zero of a continuous function. In the sequel we will show that the Conley index may be used in a similar way. To do this we need to extend the Conley index to mv maps.

5.4 Conley index for multivalued maps.

We begin with a definition of a discrete multivalued dynamical system.

Definition 5.3 An usc mv map $F : X \times \mathbf{Z} \rightrightarrows X$ with compact values is called a *discrete multivalued dynamical system (dmds)* if the following conditions are satisfied:

(i) For all $x \in X, \quad F(x,0) = \{x\}$;

(ii) For all $n, m \in \mathbf{Z}$ with $nm \geq 0$ and all $x \in X$,
$F(F(x,n),m) = F(x, n+m)$;

(iii) For all $x, y \in X, \quad y \in F(x,-1) \Longleftrightarrow x \in F(y,1)$.

It is convenient to use the notation $F^n(x) := F(x,n)$. The map F^n coincides with the superposition of n copies of $F^1 : X \rightrightarrows X$ or its inverse $(F^1)^{-1}$. This implies that the dmds F is uniquely determined by F^1, the *generator* of the dmds F. For simplicity we will denote the generator also by F.

Let $F : X \times \mathbf{Z} \rightrightarrows X$ be a dmds and let $N \subset X$ be a compact subset. Let $A \subset \mathbf{Z}$. We say that $\sigma : A \to N$ is a *solution* to F in N through x if $0 \in A$, $\sigma(0) = x$ and $\sigma(n+1) \in F(\sigma(n))$ for $n, n+1 \in A$. Define the sets

$$
\begin{aligned}
\operatorname{Inv}(N,F) &:= \{\, x \in N \mid \exists \sigma : \mathbf{Z} \to N \text{ a solution to } F \text{ in } N \text{ through } x \,\} \\
\operatorname{Inv}^{+}(N,F) &:= \{\, x \in N \mid \exists \sigma : \mathbf{Z}^{+} \to N \text{ a solution to } F \text{ in } N \text{ through } x \,\} \\
\operatorname{Inv}^{-}(N,F) &:= \{\, x \in N \mid \exists \sigma : \mathbf{Z}^{-} \to N \text{ a solution to } F \text{ in } N \text{ through } x \,\}
\end{aligned}
$$

The set N is an *isolating neighborhood* for F if

$$B_{\operatorname{diam}_N F}(\operatorname{Inv} N) \subset \operatorname{int} N,$$

where $\operatorname{diam}_N F$ is the maximal diameter of the values of F in N.

A pair $P = (P_1, P_0)$ of compact subsets $P_0 \subset P_1 \subset N$ is called an *index pair* for F in N if the following conditions are satisfied:

$$F(P_i) \cap N \subset P_i, \; i = 0, 1;$$

$$F(P_1 \backslash P_0) \subset N;$$

$$\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_0)$$

Note that unlike the single valued case we defined here the index pair relative to an isolating neighborhood. This is because of some fundamental problems we encounter when trying to define the index map. The problems come from the fact that the homotopy and homology of multivalued maps are not obvious to define. Probably the most extensively studied is the homology and cohomology theory for multivalued maps ([35, 34]). In general some assumptions, usually quite restrictive, must be imposed on the mv maps to guarantee that the map in homology makes sense. In most cases the assumptions are satisfied in applications but are lost when we want to form an analogue of the index map (5). Therefore in the multivalued case it easier to construct the index directly on the homology level. A simple class of mv maps which do induce maps in homology consists of star-shaped maps. We say that $F : X \rightrightarrows Y$ is *star-shaped* if there exists a selector $f : X \to Y$ such that for any other selector $g : X \to Y$ and any $x \in X$ the segment joining $f(x)$ and $g(x)$ is contained in $F(x)$. Notice that convex-valued maps which admit at least one selector are in particular star-shaped. In the sequel we assume that the generator F is star shaped.

For a star-shaped mv map we define the map induced in homology by

$$F_* := f_* : H_*(X) \to H_*(Y)$$

where $f : X \to Y$ is an arbitrary selector of F. The definition is correct, because F admits at least one selector and evidently any two selectors are homotopic.

If $F : (X, X_0) \rightrightarrows (Y, Y_0)$ is a star-shaped multivalued map of pairs, i.e. a mv map $F : X \rightrightarrows Y$ such that $F(X_0) \subset Y_0$, then in a similar way we define the map

$$F_* : H_*(X, X_0) \to H_*(Y, Y_0).$$

The process of collapsing the exit set to a point may expel us from the class of star-shaped maps. Therefore we have to define the index map differently. First we define *regular index pairs*. An index pair P for an isolating neighborhood $N \subset X$ is called regular if $f(U \backslash P_0) \subset P_1$ for some $U \supset P_0$ open in P_1 and

$$\mathrm{cl}(F(P_0) \backslash P_1) \cap \mathrm{cl}(P_1 \backslash P_0) = \emptyset.$$

One can show that every isolated invariant set admits regular index pairs. We put

$$T(P) : \; = \quad T_{N,F}(P) := (P_1 \cup (X \backslash \operatorname{int} N), P_0 \cup (X \backslash \operatorname{int} N)).$$

The following proposition follows from the excision property of the homology theory.

Proposition 5.4 *If P is a regular index pair for N then*

- *(i)* $F(P) \subset T_{N,F}(P)$;
- *(ii)* *The inclusion $i_{P,T(P)} : P \to T(P)$ induces an isomorphism in homology.*

Actually, the regular index pair in the above proposition may be replaced by an arbitrary index pair if one uses Alexander-Spanier cohomology instead of homology.

Consider the mv map $F_{P,T(P)} : P \ni x \to F(x) \subset T(P)$. The endomorphism $I_P := H_*(i_{P,T(P)})^{-1} \circ H_* \left(F_{P,T(P)}\right)$ of $H_*(P)$ is called the *index map* associated with the index pair P.

The following theorem lets us extend the Conley index to multivalued maps.

Theorem 5.5 *Let S be an isolated invariant set. Then the Leray reduction of $I_P : H_*(P) \to H_*(P)$ is independent of the choice of an isolating neighborhood N of S for F and a regular index pair P for N.*

Now $\mathrm{Con}_*(S, F)$, the *homological Conley index* of S is defined similarly to the single valued case as the Leray reduction of $I_P : H_*(P) \to H_*(P)$ for any regular index pair P in any isolating neighborhood N of S.

As we will see in the sequel, the importance of the generalization of the Conley index to mv maps lies in the fact that the index is inheritable. More precisely we have the following theorem.

Theorem 5.6 *The isolating neighborhood, the index pair, and the Conley index are strongly inheritable terms.*

5.5 Continuation.

As was indicated in the introduction and stated explicitly in Theorem 3.10 the Conley index continues, i.e. as long as the invariant set S remains isolated its Conley index does not change. The same is true in the mv case. To be more precise consider a compact interval $\Lambda \subset \mathbf{R}$ and

$$F : \Lambda \times X \times \mathbf{Z} \rightrightarrows X$$

be an usc star shaped map such that, for each $\lambda \in \Lambda$, $F_\lambda : X \times \mathbf{Z} \rightrightarrows X$ given by $F_\lambda(x, n) := F(\lambda, x, n)$ is a dmds. The first observation is the following easy to prove theorem.

Theorem 5.7 *Let $\lambda_0 \in \Lambda$ and let N be an isolating neighborhood for F_{λ_0}. Then N is an isolating neighborhood for F_λ for all λ sufficiently close to λ_0.*

Another way of stating Theorem 5.7 is that isolating neighborhoods are stable with respect to small perturbations of generators. To prove the stability of the Conley index itself it would be nice to have stability of index pairs. However, as pointed out in [70], this is not true in general. Never the less, in some settings stable index pairs do exist, as the following theorem shows.

Theorem 5.8 *Let $f : X \to X$ be a homeomorphism, N an isolating neighborhood for f and W an open neighborhood of* $\mathrm{Inv}\, N$. *Then there exists an index pair P for N with $P_1 \backslash P_0 \subset W$ and an $\varepsilon > 0$ such that if $G : X \to \mathcal{P}(X)$ is an usc proper map with the property*

$$G(x) \subset B_\varepsilon(f(x)), \quad \textit{for all } \; x \in X,$$

then P also is an index pair for G. In particular P is an index pair for any small perturbation of f

Observe that an immediate corollary of the above Theorem is the Continuation Property of the Conley index in the setting of homeomorphisms and flows.

5.6 Computability of the index.

Of course, in order to be able to compute the Conley index of a multivalued representation of a map we need to know how to construct index pairs. Actually, all what we need is to show that a representable mv map in a representable isolating neighborhood admits representable index pairs and find a formula for such an index pair.

This is relatively easy for special isolating neighborhoods, namely isolating blocks. A compact set $N \subset \mathbf{R}^d$ is an *isolating block* for F if

$$B_{\operatorname{diam}_N F}(F^{*-1}(N) \cap N \cap F(N)) \subset \operatorname{int} N.$$

Theorem 5.9 *If $F : X \rightrightarrows Y$ is usc representable and N is a representable isolating block for F, then N is an isolating neighborhood and*

$$(N, N \backslash F^{-1}(\operatorname{int} N))$$

is an index pair in N consisting of representable sets. Hence, index pairs may be constructed algorithmically.

Unfortunately isolating blocks are much harder to find than isolating neighborhoods. To construct an index pair for an arbitrary isolating neighborhood N we need first the following theorem in which F_N denotes the map

$$F_N : N \ni x \to N \cap F(x) \subset N.$$

Theorem 5.10 *If F and N are representable then there exist numbers $p, q \in \mathbf{N}$ such that*

$$\begin{aligned} \operatorname{Inv}^-(N, F) &= \textstyle\bigcap_{i=0,p} F_N^i(N) \\ \operatorname{Inv}^+(N, F) &= \textstyle\bigcap_{i=0,q} F_N^{*-i}(N) \end{aligned}$$

In particular $\operatorname{Inv}^- N, \operatorname{Inv}^+ N$ *and* $\operatorname{Inv} N$ *are representable.*

For a given $\hat{\mathbf{R}}$-representable set $A \subset \mathbf{R}^d$ define the following two sets

$$o_{\hat{\mathbf{R}}}(A) := \bigcup \{ E \mid E \text{ is elementary representable and} \operatorname{cl} E \cap A \neq \emptyset \}$$

$$n_{\hat{\mathbf{R}}}(A) := \bigcup \{ \operatorname{cl} E \mid E \text{ is elementary representable and} \operatorname{cl} E \cap A \neq \emptyset \}$$

One easily verifies that $o_{\hat{\mathbf{R}}}(A) \subset n_{\hat{\mathbf{R}}}(A)$ are both neighborhoods of A. Moreover, one can check that they are both representable, therefore they are algorithmically computable.

The following theorem shows how one can obtain an index pair from $\operatorname{Inv}^-(N, F)$ and $\operatorname{Inv}^+(N, F)$.

Theorem 5.11 *Assume $F : X \rightrightarrows X$ is a representable usc multivalued map and N is a representable isolating neighborhood with respect to a representation $\hat{\mathbf{R}} \subset \mathbf{R}^d$. Let $\hat{\mathbf{R}}_1$ be a refinement of $\hat{\mathbf{R}}$. Put $P_1 := n_{\hat{\mathbf{R}}_1}(\operatorname{Inv}^- N) \cap N$, $P_0 := P_1 \backslash o_{\hat{\mathbf{R}}_1}(\operatorname{Inv}^+ N)$. Then P_1, P_0 are representable with respect to $\hat{\mathbf{R}}_1$ and $P := (P_1, P_0)$ is an index pair for F in N.*

Theorem 5.11 leads to the following algorithm constructing index pairs.

Algorithm 5.12 **function** find_index_pair(N, F)
begin
$A := N; B := N;$
repeat
$A' := A; B' := B;$
$A := F_N(A); B := F_N^{*-1}(B);$
until $(A = A'$ and $B = B');$
$C := A \cap B;$
$r := \operatorname{diam}_N F;$
if $(B(C, r) \subset \operatorname{int} N)$ **then**
begin
$P_1 := n_{\hat{\mathbf{R}}_1}(A) \cap N;$
$P_0 := P_1 \setminus o_{\hat{\mathbf{R}}_1}(B);$
return $(P_1, P_0);$
end
else return "Failure";
end ;

Theorem 5.13 *If Algorithm 5.12 is called with N, a representable compact set and F, a representable multivalued dynamical system on input, then it always stops. If it does not output "Failure", then N is an isolating neighborhood for F and the algorithm outputs an index pair (P_1, P_0) for F in N.*

5.7 References

Representable multivalued maps as a tool for rigorous numerical computation of the Conley index appeared in [56, 57]. The idea of inheritance comes from [63]. The Conley index for multivalued maps was constructed in [41] and the existence of stable index pairs was established in [42]. The algorithms for computing the Conley index were presented in [63] and [80].

6 Examples of computer assisted proofs

The use of the computer to numerically approximate isolating neighborhoods and to compute the associated Conley indices represents yet another new direction in the Conley Index theory. The examples presented below are included to emphasize two points: first, it is feasible to employ the computer to obtain rigorous analysis of dynamical systems using the index theory; and two, this approach can be applied to a variety of different types of problems.

6.1 Lorenz Equations.

Representable multivalued maps together with inheritable properties open the way to rigorous computations of the Conley index for concrete dynamical systems. Since there are various qualitative descriptions of dynamics in terms of the Conley index, many new interesting results may be obtained this way. As an example consider a result on chaotic dynamics in the Lorenz equations (see [59] and also [56, 57]). The result is based on a generalization of the Conley index theory proposed by Szymczak [79].

For a $k \times k$ matrix $A = (A_{ij})$ over $\mathbf{Z}_2$ put

$$\Sigma(A) := \{\alpha : \mathbf{Z} \to \{1, 2, \ldots, k\} \mid \forall i \in \mathbf{Z} A_{\alpha(i)\alpha(i+1)} = 1\}$$

$$\sigma : \Sigma(A) \ni \alpha \to (\mathbf{Z} \ni i \to \alpha(i+1) \in \{1, 2, \ldots k\} \in \Sigma(A).$$

Let $n_{(10,28,8/3)} = 6$, $n_{(10,60,8/3)} = n_{(10,54,45)} = 4$,

$$A_{(10,28,8/3)} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

$$A_{(10,60,8/3)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad A_{(10,54,45)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

We have the following theorem

Theorem 6.1 *Consider the Lorenz equations*

$$\begin{aligned} \dot{x} &= s(y - x) \\ \dot{y} &= Rx - y - xz \\ \dot{z} &= xy - qz, \end{aligned} \tag{25}$$

and the plane $P := \{(x, y, z) \mid z = R - 1\}$. *For all parameter values in a sufficiently small neighborhood of* (σ, R, b) *there exists a Poincare section* $N \subset P$ *such that the associated Poincare map* f *is Lipschitz and well defined. Furthermore, there is a continuous map* $\rho : \mathrm{Inv}(N, f) \to \Pi_{n_{(\sigma,R,b)}}$ *such that*

$$\rho \circ f = \sigma \circ \rho$$

and $\rho(\mathrm{Inv}(N, f) \supset \Sigma(A_{(\sigma,R,b)})$. *Moreover, for each periodic* $\alpha \in \Sigma(A)$ *there exists an* $x \in \mathrm{Inv}(N, f)$ *on a periodic trajectory of the same minimal period such that* $\rho(x) = \alpha$.

An outline of the algorithm which proves the above theorem is as follows.

1. Choose a candidate for a representable isolating
neighborhood N for the Poincare map f.
2. Compute a mv representation F of f.
3. Check if N is an isolating neighborhood for F.
4. If no, find a better mv representation of f
and go back to 3.
5. If yes, compute the index pair
and find the Conley index.
6. By inheritance, f has the same Conley index in N.
7. If the computed index satisfies the assumption
of the chaos criterion, proclaim SUCCESS,
otherwise proclaim FAILURE.

6.2 The Hénon map.

Recall that the Hénon map is given by the formula

$$f(x,y) = (1 - ax^2 + y, bx).$$

A. Szymczak gave a computer assisted proof of the following result.

Theorem 6.2 *For $a = 1.4$ and $b = 0.3$ the Hénon map admits periodic orbits of all principal periods except for 3 and 5.*

The proof is based on rigorous numerical computation of the Conley index.

6.3 Kuramoto-Sivashinsky equations.

Consider the following system of ordinary differential equations with parameter λ.

$$\begin{array}{rcl} \dot{x_0} & = & x_1 \\ \dot{x_1} & = & x_2 \\ \dot{x_2} & = & -\lambda x_1 - \frac{x_0^2}{2} + 1. \end{array} \tag{26}$$

One easily verifies that the equation has exactly two stationary points: $x^- := (-\sqrt{2}, 0, 0), x^+ := (\sqrt{2}, 0, 0)$. An interesting question is if the equation admits a solution $x(t)$ such that

$$\lim_{t \to -\infty} x(t) = x^-, \quad \lim_{t \to \infty} x(t) = x^+. \tag{27}$$

In case of $\lambda = 1$ equation (26) is the ODE derived from the Kuramoto-Sivashinsky PDE and studied numerically by several authors. It is shown in [82] that for $\lambda = 1$ the equation (26) admits heteroclinic connections. In case of $\lambda = 0$ the equation becomes a variant of an equation studied by C. Conley [7], who shows that it also admits a heteroclinic connection. It turns out that combining the technique of Conley and some ideas of Troy with a computer assisted proof that certain sets are isolating neighborhoods one can show that

(26) admits a heteroclinic connection for every $\lambda \in [0,1]$. In other words the following generalization of the results of Troy and Conley may be proven.

Theorem 6.3 *For every $\lambda \in [0,1]$ there exists a solution $x_\lambda(t)$ of the equation (26) such that (27) is satisfied.*

6.4 References

The computer assisted proof of chaos in the Lorenz equations for various parameter values was presented in [56, 57, 59]. Rigorous numerical verification of Theorem 6.3 was presented in [65] and of Theorem 6.2 in [80]

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