# CONNECTED SIMPLE SYSTEMS AND THE CONLEY FUNCTOR

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ABSTRACT. We give a short and elementary proof that the Conley index is a connected simple system. We use this fact to construct the Conley functor.

## 1. INTRODUCTION.

The Conley index is a topological tool used in the qualitative theory of differential equations and dynamical systems (see [1, 10, 4]). In the simplest setting it takes the form of an object of a certain category (homotopy category of metric spaces, category of graded moduli etc.), which is known up to an isomorphism. This lack of precision is caused by the fact that there are many so called index pairs used in the construction of the index but up to an isomorphism one can extract some common information from them, which is by the definition the index. In a more subtle approach to the Conley index, essential in many applications, one shows that the isomorphisms are not quite arbitrary but form a connected simple system, i.e. a small category with the property that there is exactly one morphism between any two objects. The first proof of this fact, in case of flows, comes form Kurland [2] (see also [10, 9, 11]). All these proofs require writing down explicit and complicated formulas for the isomorphisms. Paradoxically, the proof is the only place, where the formulas seem to be necessary.

The aim of this paper is to propose an elementary proof of the fact that the Conley index is a connected simple system, which avoids writing down explicit formulas for the isomorphisms. Actually, it turns out that the inclusion relations between index pairs fully determine the structure of the Conley index as a connected simple system. In particular this means that there is no alternative way of making the Conley index a connected simple system, a fact intuitively expected but never proved.

As a by-product of our construction, we obtain the fact that the Conley index is actually a functor. This generalizes the result for flows by McCord [3] to the case of discrete dynamical systems. (The Conley index for flows is actually a special case of the Conley index for discrete semidynamical systems - see [6, Theorem 1]).

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The organisation of the paper is as follows. Section 2 contains preliminaries. Our main theorem on construction of connected simple systems comes in Section 3. Section 4 presents the construction of the Conley functor for isolating neighborhoods and the last section such a construction for isolated invariant sets.

## 2. Preliminaries.

We denote the sets of integers by $\mathbb{Z}$. Topological pairs are usually denoted by a capital letter and the corresponding two elements of the pair are marked by subscripts 1 and 2. Thus if $P$ is a topological pair then $P = (P_1, P_2)$. The standard set theoretic notation is carried over in the obvious way to topological pairs. Thus if $P, Q$ are topological pairs then $P \subset Q$ means $P_1 \subset Q_1$ and $P_2 \subset Q_2$. Similarly, $P \cap Q$ stands for the pair $P_1 \cap Q_1, P_2 \cap Q_2$.

Given a category $\mathcal{A}$, the notation $\mathrm{Obj}(\mathcal{A}), \mathrm{Mor}(\mathcal{A}), \mathrm{Iso}(\mathcal{A})$ will be used respectively for the class of objects of $\mathcal{A}$, the class of morphisms in $\mathcal{A}$ and the class of isomorphisms in $\mathcal{A}$. Given $A, B \in \mathrm{Obj}(\mathcal{A})$, the set of morphisms from $A$ to $B$ will be denoted by $\mathcal{A}(A, B)$. To simplify notation, we will often write $A \in \mathcal{A}$ instead of $A \in \mathrm{Obj}(\mathcal{A})$.

The *category of endomorphisms* of $\mathcal{A}$, denoted by $\mathrm{Endo}(\mathcal{A})$, is defined as follows. The objects of $\mathrm{Endo}(\mathcal{A})$ are pairs $(A, a)$, where $A \in \mathcal{A}$ and $a \in \mathcal{A}(A, A)$ is a distinguished endomorphism of $A$. The set of morphisms from $(A, a) \in \mathrm{Endo}(\mathcal{A})$ to $(B, b) \in \mathrm{Endo}(\mathcal{A})$ is the subset of $\mathcal{A}(A, B)$ consisting of exactly those morphisms $\varphi \in \mathcal{A}(A, B)$ for which $b\varphi = \varphi a$. We write $\varphi : (A, a) \to (B, b)$ to denote that $\varphi$ is a morphism from $(A, a)$ to $(B, b)$ in $\mathrm{Endo}(\mathcal{A})$. We define the *category of automorphisms* of $\mathcal{A}$ as the full subcategory of $\mathrm{Endo}(\mathcal{A})$ consisting of pairs $(A, a) \in \mathrm{Endo}(\mathcal{A})$ such that $a \in \mathcal{A}(A, A)$ is an automorphism, i.e., both an endomorphism and an isomorphism in $\mathcal{A}$. The category of automorphisms of $\mathcal{A}$ will be denoted by $\mathrm{Auto}(\mathcal{A})$. There is a functorial embedding

$$\mathcal{A} \ni A \to (A, \mathrm{id}_A) \in \mathrm{Auto}(\mathcal{A}), \tag{1}$$

$$\mathcal{A}(A, B) \ni \varphi \to \varphi \in \mathrm{Auto}(\mathcal{A})(A, B) \tag{2}$$

hence we can consider the category $\mathcal{A}$ as a subcategory of $\mathrm{Auto}(\mathcal{A})$.

Assume $\mathcal{C}$ is a full subcategory of $\mathrm{Endo}(\mathcal{A})$ and $F : \mathcal{C} \to \mathrm{Auto}(\mathcal{A})$ is a functor. Let $(A, a) \in \mathcal{C}$. Then $F(A, a) = (A', a')$ is an object of $\mathrm{Auto}(\mathcal{A})$. Obviously $a : (A, a) \to (A, a)$ is a morphism in $\mathrm{Endo}(\mathcal{A})$ and since $\mathcal{C}$ is a full subcategory of $\mathrm{Endo}(\mathcal{A})$ it is also a morphism in $\mathcal{C}$. Hence $F(a) : F(A, a) \to F(A, a)$ is a well defined morphism in $\mathrm{Auto}(\mathcal{A})$. However, it need not be $F(a) = a'$ in general. We say that $F : \mathcal{C} \to \mathrm{Auto}(\mathcal{A})$ is *normal*, if for each $(A, a) \in \mathcal{C}$ the morphism $F(a)$ is equal to the automorphism distinguished in $F(A, a)$. Examples of normal functors may be found in [7, 8, 12].

## 3. Connected simple systems.

Let $\mathcal{C}$ be a category and $\mathcal{E}$ its small subcategory. We will say that $\mathcal{E}$ is a *pre-connected simple system* (pre-CSS) in $\mathcal{C}$ if the following three conditions are satisfied.

$$\operatorname{Mor}(\mathcal{E}) \subset \operatorname{Iso}(\mathcal{C}) \tag{3}$$

$$\forall E_1, E_2 \in \mathcal{E} \ \operatorname{card} \mathcal{E}(E_1, E_2) \leq 1 \tag{4}$$

$$\forall E_1, E_2 \in \mathcal{E} \ \exists E_3 \in \mathcal{E} : \ \mathcal{E}(E_3, E_1) \neq \emptyset \neq \mathcal{E}(E_3, E_2) \tag{5}$$

If $E_1, E_2 \in \mathcal{E}$ and $\mathcal{E}(E_1, E_2) \neq \emptyset$ then the unique element of $\mathcal{E}(E_1, E_2)$ will be denoted by $\mathcal{E}_{E_2 E_1}$.

We will say that $\mathcal{E}$ is a *connected simple system* (CSS) in $\mathcal{C}$ if for any two objects $E_1, E_2 \in \mathcal{E}$ there exists exactly one morphism in $\mathcal{E}(E_1, E_2)$. Obviously every CSS is also a pre-CSS.

**Theorem 3.1.** *For any $\mathcal{E}$, a pre-CSS in $\mathcal{C}$, there exists a unique CSS $\bar{\mathcal{E}}$ in $\mathcal{C}$ such that*

$$\operatorname{Obj}(\mathcal{E}) = \operatorname{Obj}(\bar{\mathcal{E}}) \tag{6}$$

$$\operatorname{Mor}(\mathcal{E}) \subset \operatorname{Mor}(\bar{\mathcal{E}}) \tag{7}$$

**Proof:** We will show uniqueness first. Assume $\bar{\bar{\mathcal{E}}}$ is another such category. All we need to prove is $\operatorname{Mor}(\bar{\mathcal{E}}) \subset \operatorname{Mor}(\bar{\bar{\mathcal{E}}})$, because then the symmetric proof implies $\operatorname{Mor}(\bar{\mathcal{E}}) = \operatorname{Mor}(\bar{\bar{\mathcal{E}}})$. Thus let $\bar{\mathcal{E}}_{E_2 E_1}$ be a morphism in $\bar{\mathcal{E}}$. Using properties (6) and (5) choose $E_3 \in \mathcal{E}$ such that $\mathcal{E}(E_3, E_1) \neq \emptyset \neq \mathcal{E}(E_3, E_2)$. We have by (7)

$$\bar{\mathcal{E}}_{E_2 E_1} = \bar{\mathcal{E}}_{E_2 E_3} \bar{\mathcal{E}}^{-1}_{E_1 E_3} = \mathcal{E}_{E_2 E_3} \mathcal{E}^{-1}_{E_1 E_3} = \bar{\bar{\mathcal{E}}}_{E_2 E_3} \bar{\bar{\mathcal{E}}}^{-1}_{E_1 E_3} = \bar{\bar{\mathcal{E}}}_{E_2 E_1}$$

which means that $\bar{\mathcal{E}}_{E_2 E_1} \in \bar{\bar{\mathcal{E}}}(E_1, E_2)$. Thus uniqueness is proved.

In order to construct $\bar{\mathcal{E}}$ satisfying (6) and (7) put

$$\operatorname{Obj}(\bar{\mathcal{E}}) := \operatorname{Obj}(\mathcal{E})$$

and define the unique element of $\bar{\mathcal{E}}(E_1, E_2)$ by

$$\bar{\mathcal{E}}_{E_2, E_1} := \mathcal{E}_{E_2 E_3} \mathcal{E}^{-1}_{E_1 E_3}, \tag{8}$$

where $E_3$ is chosen so that $\mathcal{E}(E_3, E_1) \neq \emptyset \neq \mathcal{E}(E_3, E_2)$. We need to show that the definition (8) does not depend on the choice of $E_3$. Thus let $E_4$ be another object in $\mathcal{E}$ such that $\mathcal{E}(E_4, E_1) \neq \emptyset \neq \mathcal{E}(E_4, E_2)$. Choose $E_5$ such that $\mathcal{E}(E_5, E_3) \neq \emptyset \neq \mathcal{E}(E_5, E_4)$. We have then the following commutative diagram in $\mathcal{E}$

$$\begin{tikzcd}
& E_3 \arrow[dl] \arrow[dr] & \\
E_1 & E_5 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & E_2 \\
& E_4 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

which implies that

$$\mathcal{E}_{E_2E_3}\mathcal{E}^{-1}_{E_1E_3} = \mathcal{E}_{E_2E_5}\mathcal{E}^{-1}_{E_1E_5} = \mathcal{E}_{E_2E_4}\mathcal{E}^{-1}_{E_1E_4}$$

Thus the definition (8) is correct.

If $\mathcal{E}(E_1, E_2) \neq \emptyset$ then $\mathcal{E}_{E_2E_1} = \mathcal{E}_{E_2E_1}\mathcal{E}^{-1}_{E_1E_1} = \bar{\mathcal{E}}_{E_2E_1}$, which shows that (7) is satisfied. We need to prove that $\mathcal{E}$ is a category. Obviously $\bar{\mathcal{E}}_{E_1E_1} = \mathcal{E}_{E_1E_1} = \mathrm{id}_{E_1}$. To show that for any $E_1, E_2, E_3 \in \mathrm{Obj}(\mathcal{E})$ we have $\bar{\mathcal{E}}_{E_3E_1} = \bar{\mathcal{E}}_{E_3E_2}\bar{\mathcal{E}}_{E_2E_1}$ choose $E_4, E_5, E_6 \in \mathcal{E}$ such that

$$\mathcal{E}(E_4, E_1) \neq \emptyset \neq \mathcal{E}(E_4, E_2), \mathcal{E}(E_5, E_2) \neq \emptyset \neq \mathcal{E}(E_5, E_3), \mathcal{E}(E_6, E_4) \neq \emptyset \neq \mathcal{E}(E_6, E_5)$$

and consider the following commutative diagram in $\mathcal{E}$

$$\begin{tikzcd}
& & E_6 \arrow[dl] \arrow[dr] \arrow[dd] & & \\
& E_4 \arrow[dl] \arrow[dr] & & E_5 \arrow[dl] \arrow[dr] & \\
E_1 & & E_2 & & E_3
\end{tikzcd}$$

which implies

$$\begin{aligned}
\bar{\mathcal{E}}_{E_3E_2}\bar{\mathcal{E}}_{E_2E_1} = \mathcal{E}_{E_3E_5}\mathcal{E}^{-1}_{E_2E_5}\mathcal{E}_{E_4E_2}\mathcal{E}^{-1}_{E_4E_1} = \\
\mathcal{E}_{E_3E_5}\mathcal{E}_{E_5E_6}\mathcal{E}^{-1}_{E_4E_6}\mathcal{E}^{-1}_{E_1E_4} = \mathcal{E}_{E_3E_6}\mathcal{E}^{-1}_{E_4E_6} = \bar{\mathcal{E}}_{E_3E_1} \quad (9)
\end{aligned}$$

□

We define the *category of connected simple systems* over $\mathcal{C}$, denoted $\mathrm{CSS}(\mathcal{C})$, as follows. We take all connected simple systems in $\mathcal{C}$ as objects of this category. If $\mathcal{E}$ and $\mathcal{F}$ are two connected simple systems in $\mathcal{C}$ then by a morphism from $\mathcal{E}$ to $\mathcal{F}$ we mean any collection

$$\varphi := \{\varphi_{FE} \in \mathcal{C}(E, F) \mid E \in \mathcal{E}, F \in \mathcal{F}\}$$

of morphisms in $\mathcal{C}$ which satisfy

$$\varphi_{F'E'} = \mathcal{E}_{F'F}\varphi_{FE}\mathcal{E}_{EE'}$$

for any $E, E' \in \mathcal{E}, F, F' \in \mathcal{F}$. The elements of $\varphi$ will be called representants of $\varphi$.

If $\psi := \{\psi_{GF} \in \mathcal{C}(F, G) \mid F \in \mathcal{F}, G \in \mathcal{G}\}$ is a morphism from $\mathcal{F}$ to $\mathcal{G} \in \mathrm{CSS}(\mathcal{C})$ then it is straightforward to verify that for given objects $E \in \mathcal{E}, G \in \mathcal{G}$ the composition

$\psi_{GF}\varphi_{FE}$ does not depend on the choice of an object $F \in \mathcal{F}$. Thus the morphism $(\psi\varphi)_{FE} := \psi_{GF}\varphi_{FE}$ is well defined and we can set

$$\psi\varphi := \{(\psi\varphi)_{GE} \mid E \in \mathcal{E}, G \in \mathcal{G}\}.$$

The commutativity of the diagram

$$\begin{array}{ccccc} E & \overset{\varphi_{FE}}{\to} & F & \overset{\psi_{GF}}{\to} & G \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ E' & \overset{\varphi_{F'E'}}{\to} & F' & \overset{\psi_{G'F'}}{\to} & G' \end{array}$$

implies that $\psi\phi$ is a well defined composition of morphisms. It is now straightforward to verify that $\{\mathcal{E}_{E'E} \mid E, E' \in \mathrm{Obj}(\mathcal{E})\}$ is the identity morphism in $\mathcal{E}$. Thus we have proved

**Theorem 3.2.** $\mathrm{CSS}(\mathcal{E})$ *is a category.*

The following proposition is an easy excercise.

**Proposition 3.3.** *A morphism of connected simple systems is an isomorphism iff all its representants are isomorphisms or equivalently if it admits at least one representant which is an isomorphism.*

Assume $\mathcal{E}, \mathcal{F} \in \mathrm{CSS}(\mathcal{C})$ and $E \in \mathcal{E}, F \in \mathcal{F}$. If $\kappa \in \mathcal{C}(E, F)$ then one can easily verify that

$$\kappa_{\mathcal{F}\mathcal{E}} := \{\, \kappa_{F'E'} \mid E' \in \mathcal{E}, F' \in \mathcal{F}\,\},$$

where

$$\kappa_{F'E'} := \mathcal{E}_{FF'}\kappa\mathcal{E}_{EE'} \text{ for } E' \in \mathcal{E}, F' \in \mathcal{F},$$

defines a morphism in $\mathrm{CSS}(\mathcal{C})(\mathcal{E}, \mathcal{F})$.

The following proposition is straightforward.

**Proposition 3.4.** *Assume* $E, E' \in \mathcal{E}, F, F' \in \mathcal{F}$ *and* $\kappa \in \mathcal{C}(E, F), \lambda \in \mathcal{C}(E', F')$. *Then* $\kappa_{\mathcal{F}\mathcal{E}} = \lambda_{\mathcal{F}\mathcal{E}}$ *iff the following diagram commutes*

$$\begin{array}{ccc} E & \overset{\kappa}{\to} & F \\ \big\downarrow \mathcal{E}_{EE'} & & \big\downarrow \mathcal{E}_{FF'} \\ E' & \overset{\lambda}{\to} & F' \end{array}$$

## 4. Isolating neighborhoods

The *category of discrete semidynamical systems* DS (briefly: the category of dynamical systems) is defined as follows. Its objects are pairs of the form $(X, f)$, where $X$ is a locally compact metric space and $f : X \to X$ is a continuous map. If $(X, f), (Y, g) \in$ DS then $\varphi : (X, f) \to (Y, g)$ is a morphism in DS if $\varphi : X \to Y$ is a partial continuous map such that $\varphi f(x) = g\varphi(x)$ for $x \in \operatorname{dom}\varphi \cap f^{-1}(\operatorname{dom}\varphi)$.

Given $(X, f) \in$ DS and $N$, a compact subset of $X$, we define the invariant part of $N$ with respect to $f$ as follows

$$\operatorname{Inv}(N, f) := \operatorname{Inv} N := \\ \{\, x \in N \mid \exists \sigma : \mathbb{Z} \to N \ \sigma(0) = x \text{ and } \forall i \in \mathbb{Z}\ f(\sigma(i)) = \sigma(i+1) \,\} \quad (10)$$

We say that $N$ is an *isolating neighborhood* if $\operatorname{Inv} N \subset \operatorname{int} N$.

**Proposition 4.1.** *If $\varphi : (X, f) \to (Y, g)$ is a morphism in* DS *and $M$ is an isolating neighborhood with respect to $g$ then $\varphi^{-1}(M)$ is an isolating neighborhood woth respect to $f$.*

**Proof:** We have

$$\operatorname{Inv}(\varphi^{-1}(M), f) \subset \varphi^{-1}(\operatorname{Inv}(M, g)) \subset \varphi^{-1}(\operatorname{int} M) \subset \varphi^{-1}(M)$$

□

The *category of isolating neighborhoods*, denoted IN, is defined as follows. The objects of IN are triples $(X, f, N)$, where $(X, f) \in$ DS and $N$ is an isolating neighborhood with respect to $f$. If $(X, f, N), (Y, g, M) \in$ IN then $\varphi : (X, f, N) \to (Y, g, M)$ is a morphism in IN if $\varphi : (X, f) \to (Y, g)$ is a morphism in DS such that $\operatorname{dom}\varphi \supset N$ and $\varphi^{-1}(M) = N$. It is an easy excercise to verify that IN is indeed a category.

The pair $P = (P_1, P_2)$ of compact subsets of $N$ is called an *index pair* in $N$ (with respect to $f$) if the following three conditions are satisfied.

$$P_i \cap f^{-1}(N) \subset f^{-1}(P_i) \text{ for } i = 1, 2 \quad (11)$$

$$P_1 \backslash P_2 \subset f^{-1}(N) \quad (12)$$

$$\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2). \quad (13)$$

The collection of all index pairs in $N$ with respect to $f$ will be denoted by $\operatorname{IP}(X, f, N)$.

The following theorem may be found in [8, 9, 5]

**Theorem 4.2.** *If $(X, f) \in$ DS and $N$ is an isolating neighborhood with respect to $f$ then for any $U$, an open neighborhood of $\operatorname{Inv} N$, there exists an index pair $P$ in $N$ such that $P_1 \setminus P_2 \subset U$. In paritcular $\operatorname{IP}(X, f, N) \neq \emptyset$.*

The following proposition is an easy excercise.

**Proposition 4.3.** *If $P, Q$ are index pairs in $N$, then also $P \cap Q := (P_1 \cap Q_1, P_2 \cap Q_2)$ is an index pair in $N$.*

Let $P_1/P_2$ denote the quotient space. We will consider it as an object in $\mathrm{Comp}_*$, the category of pointed metric compact spaces, by assuming that the point distinguished in $P_1/P_2$ is $P_2$ collapsed to a point. We will denote this point $[P_2]$.

It follows easily from the definition of the index pair that the map $f$ induces a continuous map $f_P : P_1/P_2 \to P_1/P_2$ given by

$$f_P(x) := \begin{cases} [f(x)] & \text{if } x \in P_2 \\ [P_2] & \text{otherwise} \end{cases}$$

It is called the *index map*. Thus we have $(P, f_P) \in \mathrm{Endo}(\mathrm{Comp}_*)$. We will write briefly $P_f := (P, f_P)$.

The proof of the following proposition is left to the reader as an easy excercise.

**Proposition 4.4.** *Assume* $(X, f, N), (Y, g, M) \in \mathrm{IN}$ *and* $P \in \mathrm{IP}(X, f, N), Q \in \mathrm{IP}(Y, g, M)$. *If* $\varphi : (X, f, N) \to (Y, g, M)$ *is a morphism in* IN *such that* $\varphi(P) \subset Q$, *then*

$$\varphi_{PQ} : P_1/P_2 \ni [x] \to [\varphi(x)] \in Q_1/Q_2$$

*is a well defined, continuous map such that the following diagram commutes*

$$\begin{array}{ccc} P_1/P_2 & \stackrel{f_P}{\to} & P_1/P_2 \\ \Big\downarrow \varphi_{PQ} & & \Big\downarrow \varphi_{PQ} \\ Q_1/Q_2 & \stackrel{g_Q}{\to} & Q_1/Q_2 \end{array}$$

Assume now that $K : \mathrm{Comp}_* \to \mathcal{A}$ is a given co- or contra-variant functor, which is homotopy invariant. It extends in a natural way to a functor $K : \mathrm{Endo}(\mathrm{Comp}_*) \to \mathrm{Endo}(\mathcal{A})$ denoted with the same letter. Assume also that $\mathcal{C} \subset \mathrm{Endo}(\mathrm{Comp}_*)$ is a subcategory such that $K(\mathrm{Endo}(\mathrm{Comp}_*)) \subset \mathcal{C}$. Let $L : \mathcal{C} \to \mathrm{Auto}(\mathcal{A})$ be a given normal functor. Then the composite functor $LK := L \circ K : \mathrm{Endo}(\mathrm{Comp}_*) \to \mathrm{Auto}(\mathcal{A})$ is defined.

**Theorem 4.5.** *(see* [8, Theorem 6.3]*). Assume* $N$ *is an isolating neighborhood with respect to* $f$, $P \subset Q$ *are two index pairs in* $N$ *and* $\iota_{PQ}$ *is the corresponding inclusion map. Then* $\iota_{PQ} \in \mathrm{Endo}(\mathrm{Comp}_*)(P_f, Q_f)$ *and* $LK(\iota_{PQ})$ *is an isomorphism in* $\mathrm{Auto}(\mathcal{A})$.

We define the category $\mathrm{Con}_{K,L}(X, f, N)$ as follows.

$$\mathrm{Obj}(\mathrm{Con}_{K,L}(X, f, N)) := \{\, LK(P_f) \mid P \in \mathrm{IP}(X, f, N)\,\} \tag{14}$$

$$\mathrm{Con}_{K,L}(X, f, N)(LK(P_f), LK(Q_f)) := \begin{cases} LK(\iota_{PQ}) & \text{if } P \subset Q \\ \emptyset & \text{otherwise} \end{cases} \tag{15}$$

Since the functors $K, L$ are fixed, we will drop subscripts $K, L$ in the sequel.

The following proposition follows easily from 4.3

**Proposition 4.6.** Con$(X, f, N)$ *is a pre*-CSS.

Thus, by Theorem 3.1, Con$(X, f, N)$ extends in a unique way to a CSS, called the *Conley index* of $f$ in $N$. We will denote this CSS again by Con$(X, f, N)$.

In order to turn Con into a functor we need the following proposition.

**Proposition 4.7.** *Assume* $(X, f, N), (Y, g, M)$ *are two objects in* IN *and* $\varphi : (X, f, N) \to (Y, g, M)$ *is a morphism in* IN*. If* $Q \in \mathrm{IP}(Y, g, M)$*, then* $\varphi^{-1}(Q) := (\varphi^{-1}(Q_1), \varphi^{-1}(Q_2)) \in \mathrm{IP}(X, f, N)$.

**Proof:** Assume $x \in \varphi^{-1}(Q_i) \cap f^{-1}(N)$. Then $\varphi(x) \in Q_i$ and $g(\varphi(x)) \in M$. It follows from (11) that $\varphi(f(x)) = g(\varphi(x)) \in Q_i$, i.e. $f(x) \in \varphi^{-1}(Q_i)$. This proves (11) for $\varphi^{-1}(Q)$. Assume in turn that $x \in \varphi^{-1}(Q_1) \setminus \varphi^{-1}(Q_2)$. Then $\varphi(x) \in Q_1 \setminus Q_2 \subset g^{-1}(M)$, i.e. $x \in \varphi^{-1}(g^{-1}(M)) = f^{-1}(N)$. This proves (12). Finally observe that

$$\mathrm{Inv}(\varphi^{-1}(M), f) \subset \varphi^{-1}(\mathrm{Inv}(M, g)) \subset \varphi^{-1}(\mathrm{int}(Q_1 \setminus Q_2)) \subset \mathrm{int}(\varphi^{-1}(Q_1) \setminus \varphi^{-1}(Q_2)),$$

which proves (13). □

Assume now that $(X, f, N), (Y, g, M) \in$ IN and $\varphi : (X, f, N) \to (Y, g, M)$ is a morphism in IN. Let $Q \in \mathrm{IP}(Y, g, M)$. Then, by Proposition 4.7, $\varphi^{-1}(Q) \in \mathrm{IP}(X, f, N)$ and since $\varphi(\varphi^{-1}(Q)) \subset Q$, we have a well defined morphism $\varphi_Q := \varphi_{\varphi^{-1}(Q)Q} : \varphi^{-1}(Q)_f \to Q_g$. This morphism in turn gives rise to the morphism $LK(\varphi_Q)_{\mathrm{Con}(X,f,N),\mathrm{Con}(Y,g,M)} : \mathrm{Con}(X, f, N) \to \mathrm{Con}(Y, g, M)$. We will show that this morphism is independent of the choice of the index pair $Q$. Let $P \in \mathrm{IP}(Y, g, M)$ be another index pair. Assume first that $P \subset Q$. Then one easily verifies that the following diagram commutes

$$\begin{array}{ccc}
\varphi^{-1}(P)_f & \stackrel{\varphi_P}{\to} & P_g \\
\Big\downarrow \iota_{\varphi^{-1}(P)\varphi^{-1}(Q)} & & \Big\downarrow \iota_{PQ} \\
\varphi^{-1}(Q)_f & \stackrel{\varphi_Q}{\to} & Q_g
\end{array}$$

Applying the functor $LK$ to the above diagram we obtain a commutative diagram in Auto$(\mathcal{A})$ which shows, by Proposition 3.4 that

$$LK(\varphi_Q)_{\mathrm{Con}(X,f,N),\mathrm{Con}(Y,g,M)} = LK(\varphi_P)_{\mathrm{Con}(X,f,N),\mathrm{Con}(Y,g,M)}.$$

If $P \not\subset Q$ then $R := P \cap Q$ is an index pair conatained both in $P$ and $Q$ and consequently

$$LK(\varphi_Q)_{\mathrm{Con}(X,f,N),\mathrm{Con}(Y,g,M)} = LK(\varphi_R)_{\mathrm{Con}(X,f,N),\mathrm{Con}(Y,g,M)} = LK(\varphi_P)_{\mathrm{Con}(X,f,N),\mathrm{Con}(Y,g,M)}. \quad (16)$$

Thus we can put

$$\mathrm{Con}(\varphi) := LK(\varphi_Q)_{\mathrm{Con}(X,f,N),\mathrm{Con}(Y,g,M)},$$

where $Q$ is any index pair for $g$ in $M$. It is now an elementary task to verify the following theorem

**Theorem 4.8.** Con : IN $\to$ Auto($\mathcal{A}$) *is a well defined functor.*

We will call it the *Conley functor*.

## 5. Isolated invariant sets.

The set $S \subset X$ is said to be *invariant* if $f(S) = S$. This is easily seen to be equivalent to $S = \operatorname{Inv}(S, f)$. $S$ is called an *isolated invariant set*, if it admits a compact neighborhood $N$ such that $S = \operatorname{Inv} N$. The neighborhood $N$ is then called an isolating neighborhood of $S$. The family of all isolating neighborhoods for $S$ will be denoted by $\mathrm{IN}(X, f, S)$.

The following proposition is straightforward.

**Proposition 5.1.** *If* $N, M \in \mathrm{IN}(X, f, S)$ *then* $N \cap M \in \mathrm{IN}(X, f, S)$

The *category of isolated invariant sets*, denoted IIS is defined as follows. The objects of IIS are triples $(X, f, S)$, where $(X, f) \in \mathrm{DS}$ and $S$ is an isolated invariant set with respect to $f$. If $(X, f, S), (Y, g, T)$ are two objects in IIS then $\varphi : (X, f, S) \to (Y, g, T)$ is a morphism in IIS if $\varphi : (X, f) \to (Y, g)$ is a morphism in DS and $\varphi^{-1}(T) = S$. It is an easy excercise to verify that IIS is indeed a category.

Note that unlike the case of an isolating neighborhood, the inverse image of an isoalted invariant set need not be an isolated invariant set. Nevertheless we have the following proposition.

**Proposition 5.2.** *Assume* $\varphi : (X, f, S) \to (Y, g, T)$ *is a morphism in* IIS *and* $M$ *is an isolating neighborhood for* $T$. *Then* $\varphi^{-1}(M)$ *is an isolating neighborhood for* $S$.

**Proof:** Since $S = \varphi^{-1}(T) \subset \varphi^{-1}(M)$, we have

$$S = \operatorname{Inv}(S, f) \subset \operatorname{Inv}(\varphi^{-1}(M), f) \subset \varphi^{-1}(\operatorname{Inv}(M, g)) = \varphi^{-1}(T) = S.$$

Thus $S = \operatorname{Inv}(\varphi^{-1}(M), f)$. $\square$

Assume now that $N, M \in \mathrm{IN}(X, f, S)$ are two isolating neighborhoods such that $M \subset N$. Then the inclusion $\iota : M \to N$ induces a map $\iota : (X, f, M) \to (X, f, N)$ and consequently we have a morphism $LK(\iota) : \operatorname{Con}(X, f, M) \to \operatorname{Con}(X, f, N)$. Select an index pair $Q \in \mathrm{IP}(X, f, N)$ such that $Q_1 \setminus Q_2 \subset \operatorname{int} M$ (such a choice is possible by Theorem 4.2). Put $P := \iota^{-1}(Q)$. It is straightforward to verify that $\iota_{PQ} : P_1/P_2 \to Q_1/Q_2$ is a homeomorphism. It follows that $LK(\iota_{PQ})$ is an isomorphism and by Proposition 3.3 also $LK(\iota)_{\operatorname{Con}(X,f,M)\operatorname{Con}(X,f,N)} : \operatorname{Con}(X, f, M) \to \operatorname{Con}(X, f, N)$ is an isomorphism.

We define the category $\operatorname{Con}(X, f, S)$ as follows.

$$\operatorname{Obj}(\operatorname{Con}(X, f, S)) := \{ \operatorname{Con}(X, f, N) \mid N \in \mathrm{IN}(X, f, S) \}$$

$$\mathrm{Con}(X,f,S)(\mathrm{Con}(X,f,M),\mathrm{Con}(X,f,N)) := \begin{cases} LK(\iota)_{\mathrm{Con}(X,f,M)\,\mathrm{Con}(X,f,N)} & \text{if } M \subset N \\ \emptyset & \text{otherwise} \end{cases}$$

The following proposition follows easily from Proposition 5.1

**Proposition 5.3.** $\mathrm{Con}(X,f,S)$ *is a pre-*CSS*.*

Thus, by Theorem 3.1, $\mathrm{Con}(X,f,S)$ extends in a unique way to a CSS, called the Conley index of $S$ with respect to $f$. We will denote this CSS again by $\mathrm{Con}(X,f,S)$.

Now assume that $\varphi : (X,f,S) \to (Y,g,T)$ is a morphism in IIS and choose $M \in \mathrm{IN}(Y,g,T)$. By Proposition 4.1 $N := \varphi^{-1}(M) \in \mathrm{IN}(X,f,S)$, hence we have a morphism $LK(\varphi)_{\mathrm{Con}(X,f,N)\,\mathrm{Con}(Y,g,M)}$. Proceeding smilarly to the case of isolating neighborhood we conclude that this morphism does not depend of the choice of the isolating neighborhood $M$. Thus we may put

$$\mathrm{Con}(\varphi) := \left(LK(\varphi)_{\mathrm{Con}(X,f,N)\,\mathrm{Con}(Y,g,M)}\right)_{\mathrm{Con}(X,f,S)\,\mathrm{Con}(Y,g,T)}$$

A routine verifications leads to the following theorem

**Theorem 5.4.** $\mathrm{Con} : \mathrm{IIS} \to \mathrm{Auto}(\mathcal{A})$ *is a well defined functor.*

Thus we again obtain a Conley functor, this time for isolated invariant sets.

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