RIGOROUS NUMERICS OF CHAOTIC DYNAMICAL SYSTEMS

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1. Introduction

The importance of computers in the present day research on dynamical systems cannot be questioned. Todays physics or chemistry would not function without access to permanently faster and larger computers. However, from the mathematical point of view, the role of computers is still very limited in this process. Of course, mathematics provides numerical algorithms for solving differential equations which gover the respective dynamical systems. By means of these algorithms computers find discrete models of the equations. Standard numerical analysis also tells us how much the numerically computed solutions in the discrete model differ over a given compact time-space domain from the real solutions. This is usually suplemented by a theorem which says that the solutions of the discrete model limit to some real solutions with the complexity of the discrete model (size of grid discretizing the space) going to infinity. Classical numerical analysis essentially does not go beyond this point. This is maybe sufficient for an ingenieur interested in approximating one particular solution representing the physical object he wants to construct. However, when global understanding of dynamics is of interest such an approach is definitely not satisfactory for at least two reasons. First of all, it does not answer the question to what extend the global asymptotic properties of the constructed discrete model reflect the properties of the original equation. One can construct examples where global phenomena present in all discrete approximations disappear in the limit. Secondly, even assuming that the computations on real numbers are performed exactly, one can only study the behaviour of solutions of the discretized model over some finite time period, which always may differ from the asymptotic behaviour. Thus the physicist or chemist is left to decide on his own, by means of his intuition and experience, if the discrete model he uses appropriately reflects the dynamics of his interest. I do not want to say that physicists fail in this respect. On the contrary, I am pretty sure they are very good in their judgments. But I do not believe they are totally uninterested in making rigorous the claims they draw from the computer simulations of the dicrete models about the asymptotic behaviour of the solutions of the equation. They only got used to the fact that the analytic methods of classical mathematics cannot tell much about global dynamics of concrete equations.

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The purpose of this paper is to show that things start to change in this respect due to a proper combination of analytic and computer based methods. First of all let us mention that recently a new field emerges: the dynamics of numerics. The aim of this field is to give criteria under which one can claim that the qualitative properties of discrete models are preserved in limit. Whenever the criteria are satisfied, one knows that if only his discrete model is sufficiently close to the original equation then its asymptotic behaviour adequately reflects the behaviour of the equation. The interested reader is referred to [3, 4, 19] for information on the achievements of the new field.

The methods of dynamics of numerics constitute an enormous step forward but do not solve fully the problem. Namely, one no longer needs to worry about carrying over the qualitative properties of the discrete model to the original equations if only one knows that the discrete model is sufficiently close to the equation. However, the question whether the model is sufficiently close may be answered again by experience only.

An alternative approach is provided by the so called computer assisted proofs. Roughly speaking mathematical conditions fall into two categories: verifiable in a finite direct computation like an inequality among two rationals or nonverifiable in a finite direct computation like the existence of a zero of a polynomial. Since the conditions of interest in dynamics usually belong to the other class, in order to successfully perform a computer assisted proof in dynamics one needs a theorem which converts verifiable conditions into nonverifiable conditions, i.e. a theorem which has some verifiable conditions as assumptions and the nonverifiable conditions of interest as assertions.

Let us explain the feasibility of such an approach on a problem which actually is the easiest one among the problems solvable analytically but just for that reason it illustrates in the easiest way the ideas we want to present. Namely consider the problem of the existence of stationary points of a scalar differential equation. It reduces to finding zeros of the vector field. Since in general one can expect the zeros to be irrational numbers, it is unlikely to find them in a finite computation via verifying some clever guesses. The zeros may be found approximately by the Newton method for instance, but the method, even if the successive iterates behave like a convergent sequence, does not guarantee that a zero indeed exists. The standard analytic approach is to look for two points where the vector field has opposite signs and then to use the Darboux property to establish the existence of a zero. If determining analytically the signs of the vector field at a given point is hard then we can use a numerical procedure to determine the signs. This may be done in a finite computation and rigorously if rigorous error estimates are included. Thus a theorem which converts the verifiable conditions into the existence of a zero of a vector field may be the Darboux property.

Via the shooting methods the Darboux property may be fruitfully applied to computer assisted proofs of the existence of phenomena much harder to achieve analytically then just the existence of stationary points. A good example is the proof of the existence of parameter values of the Lorenz equations for which there exists a trajectory homoclinic to the origin. An analytic proof of this fact based on the shooting methods was proposed by Hastings and Troy [6] and later Hassard and Zhang [7] gave a computer assisted proof of this fact which provided very narrow bounds for the parameter admitting the homoclinic trajectory. Similar methods where used to show complicated behaviour of solutions of the Lorenz equations in [8]. Let us also mention two other interesting computer assisted proofs by Neumaier and Rage [17] and also by Rage, Neumaier and Schlier [18]. In the first paper they prove chaos in the standard map. In the other chaos is obtained in a Hamiltonian system govering a molecular model. In both cases the proof consists in verifying in a rigorous computation the existence of a transversal homoclinic point.

2. Multivalued maps and inheritable properties

The approach we want to propose is based on finite multivalued dynamics and the concept of inheritable properties. Multivalued upper- or lower semicontinuous maps which admit only a finite set of values provide a natural bridge between the uncountable dynamics of differential equations and the finite dynamics treatable by computer. The point is that on one hand properties of such maps are often verifiable in a finite computation and on the other hand many such properties are inherited by single valued continuous selectors of such maps.

In order to be more concrete let us introduce some definitions. First of all observe that only a finite set of numbers $\hat{\mathbb{R}} \subset \mathbb{R}$ can be used in computer arithmetic. We will call the elements of $\hat{\mathbb{R}}$ *representable numbers*. Since for obvious reasons the arithmetic operations on representable numbers cannot be exact, some rounding tracking mechanism is needed. The most elegant approach is known as the interval arithmetic (comp. [1]). Assume A is a subset of $\mathbb{R}$. The set of *intervals over* A is defined as

$$\mathcal{I}(A) := \{[a,b] \mid a,b \in A, a \leq b\}$$

We treat the set of real numbers as a subset of $\mathcal{I}(\mathbb{R})$ via the following natural embedding

$$\mathbb{R} \ni x \to [x,x] \in \mathcal{I}(\mathbb{R})$$

Let $\diamond \in \{+,-,*,/\}$. We extend the arithmetic operations to $I, J \in \mathcal{I}(\mathbb{R})$ by

$$I \diamond J := \{a \diamond b \mid a \in I, b \in J\}$$

The intervals over the set of representable numbers will be called *representable intervals*. By the *representation* of a real number $x \in \mathbb{R}$ we mean any representable interval containing x. Similarly we define the *representation* of an interval $I \in \mathcal{I}(\mathbb{R})$. The *optimum representation* of I is given by

$$\updownarrow I := \bigcap\{J \in \mathcal{I}(\hat{\mathbb{R}}) \mid I \subset J\}$$

We define the adjusted arithmetic operations on representable intervals $I, J \in \mathcal{I}(\hat{\mathbb{R}})$ by

$$I \mathbin{\hat{\diamond}} J := \updownarrow(I \diamond J)$$

Note that the adjusted arithmetic operations on representable intervals are not associative.

Assume $Q : \mathbb{R}^p \to \mathbb{R}$ is a polynomial. By an *implementation* of Q we mean a function $\mathcal{Q} : \hat{\mathbb{R}}^p \to \mathcal{I}(\hat{\mathbb{R}})$ obtained by replacing every arithmetic operation in Q by the corresponding adjusted arithmetic operation in $\mathcal{I}(\hat{\mathbb{R}})$.

Note: A polynomial may have several implementations depending on the order of evaluation of the elementary arithmetic operations.

Proposition 2.1. *Assume $\mathcal{Q}$ is an implementation of a polynomial Q. Then*

$$\forall (x_1, x_2, ...x_p) \in \mathbb{R}^p \; Q(x_1, x_2, ...x_p) \in \mathcal{Q}(x_1, x_2, ...x_p)$$

The interval arithmetic allows us to find rigorously multivalued maps enclosing the exact maps of our interest. The procedure is as follows. First we define the collection of *elementary representable sets* as the family of all singletons $\{a\}$, where $a \in \hat{\mathbb{R}}$, and all open intervals (a, b), where $a, b \in \hat{\mathbb{R}}$ are such that b is an immediate successor of a in $\hat{\mathbb{R}}$. The Cartesian product of d elementary representable sets will be called an *elementary representable set*. A *representable set* is defined as the union of a collection of elementary representable sets. Let $A \subset \mathbb{R}^p$. A multivalued map $F : A \to 2^{\mathbb{R}^q}$ will be called *representable* if $A = \operatorname{dom} F$ and the values of F are representable and F restricted to an elementary representable set is constant. Now let $U \subset \mathbb{R}^p$ and let $f : U \to \mathbb{R}^q$ be an L-Lipschitz continuous single valued map. We will say that the multivalued map F is a *representation of f* if F is representable and f is a selector of $F_{|U}$, i.e. $U \subset \operatorname{dom} F$ and for every $x \in U$ we have $f(x) \in F(x)$. A multivalued representation of f may computed in the following way. First of all a finite collection $\mathcal{B}$ of representable sets covering U is chosen. Then a point x_B is selected in every representable set $B \in \mathcal{B}$ and an approximate value of the map f at the point is found. Now an appropriate error analysis (reducing to to the interval computations if f is a polynomial) allows one to find a representable set containing the exact value $f(x_B)$. Using the Lipschitz constant one finds a representable set $\mathcal{F}_B$ such that $f(B) \subset \mathcal{F}_B$. One then defines the following two maps

$$F^- : \bigcup \mathcal{B} \ni x \to \bigcap \{\, \mathcal{F}_B \mid x \in B \in \mathcal{B} \,\} \in \mathcal{P}(\mathbb{R}^q)$$

$$F^+ : \bigcup \mathcal{B} \ni x \to \bigcup \{\, \mathcal{F}_B \mid x \in B \in \mathcal{B} \,\} \in \mathcal{P}(\mathbb{R}^q)$$

It is easy to verify that both F^-, F^+ are representations of f and additionally F^- is lower semicontinuous and F^+ is upper semicontinuous. Moreover, such multivalued representations may be chosen arbitrarily close to f if only a sufficiently powerful computer is available.

The importance of representable mulltivalued maps lied in the fact, that most their properties may be verified in a finite, i.e. combinatorial computation.However we need some mechanism to transfer such properties to single valued selectors. For that reason we introduce the following notion. Assume $\mathcal{A}$ is a collection of mv maps and $\varphi(F)$ is a property of such maps. We will say that φ is *inheritable* (comp. [14]) if the fact that $\varphi(F)$ holds for some multivalued map $F \in \mathcal{A}$ implies that it holds for every single-valued selector of F in $\mathcal{A}$. An inheritable property φ will be called *strongly inheritable* if for any single valued map $f \in \mathcal{A}$ such that $\varphi(f)$ and for any sequence $\{F_n\} \subset \mathcal{A}$ satisfying $F_n \to f$ we have $\varphi(F_n)$ for n sufficiently large.

Here are some examples. Take $\mathcal{A} = \{F : \mathbb{R}^p \to 2^{\mathbb{R}}\}$ and consider the following three properties

$$\begin{aligned} \alpha(F) &\Leftrightarrow \exists x \in \mathbb{R}^p\ F(x) > 0 \\ \beta(F) &\Leftrightarrow \exists x \in \mathbb{R}^p\ F(x) \geq 0 \\ \gamma(F) &\Leftrightarrow \exists x \in \mathbb{R}^p\ F(x) \ni 0 \end{aligned}$$

The first property is strongly inheritable. The second property is inheritable but not strongly inheritable. The last property is not inheritable.

Assume $\mathcal{A} = \{F : \mathbb{R}^p \to 2^{\mathbb{R}^p}\}$ and let B be the unit ball in $\mathbb{R}^p$. Consider the following three properties

$$\begin{aligned} \mu(F) &\Leftrightarrow F(B) \subset \operatorname{int} B \\ \nu(F) &\Leftrightarrow F(B) \subset B \\ \kappa(F) &\Leftrightarrow \exists x \in B\ x \in F(x) \end{aligned}$$

Again the first property is strongly inheritable, the second property is inheritable but not strongly inheritable and the last property is not inheritable.

We will explain the importance of the inheritable properties in the following sections.

3. A geometric criterion of chos.

Let (X, X_0) be a pair of topological spaces. Let

$$\Gamma(X, X_0) := \{\, [\xi] \mid \xi : ([0,1], \{0,1\}) \to (X, X_0) \,\}$$

denote the family of homotopy classes of arcs in X with endpoints in X_0.

We will say that (A, A_0) is an I-pair if it is homotopically equivalent to the pair $([0,1], \{\, 0,1 \,\})$. It is easy to verify that if (A, A_0) is an I-pair then A is contractible and A_0 has exactly two arcwise connected components, which we will denote by A_0^-, A_0^+. Moreover, if $\xi_0, \xi_1 \in \Gamma(A, A_0)$ then $[\xi_0] = [\xi_1]$ iff $\xi_0(0), \xi_1(0)$ and $\xi_0(1), \xi_1(1)$ are in the same components of A_0. In particular $\Gamma(A, A_0)$ has exactly four elements $[\kappa_{i,j}]$ for $i, j \in \{0, 1\}$, where $\kappa_{i,j} : ([0,1], \{0,1\}) \to (A, A_0)$ is an arc such that $\kappa_{i,j}(0) \in A_0^-$ and $\kappa_{i,j}(1) \in A_0^+$.

Let $J_0 := [-2, -1]$, $J_1 := [1, 2]$, $J := J_0 \cup J_1$. For $i = 0, 1$ define the map

$$\mu_i : J_i \ni s \to 6s + (-1)^i 9 \in \mathbb{R}$$

and put $\mu := \mu_0 \cup \mu_1 : J \to \mathbb{R}$. Assume that $N = T \times J$, where T is a contractible space. Put $N^h := T \times \{-2, -1, 1, 2\}$, $N^v := \mathrm{cl}(\mathrm{bd}\, N \setminus N^h)$, $N_i := T \times J_i$, $N_i^h := T \times \mathrm{bd}\, J_i$, $N_i^\pm := T \times \mathrm{bd}^\pm J_i$, where $\mathrm{bd}\, J_i$ stands for the doubleton consisting of the endpoints of J_i and $\mathrm{bd}^\pm J_i$ denote the singletons consisting of the left and right endpoints of J_i respectively. For a point $c \in T$ and $i = 0, 1$ define a map $\mu_i^c : ([0,1], \{0,1\}) \to (X \setminus N^v, X \setminus N)$ by $\mu_i^c(t) := (c, \mu_i(t))$.

The following theorem is a variant of a result by Zgliczy/'nski (comp. [14, 21]).

Theorem 3.1. *Assume $f : X \to X$ is a homeomorphism and there exists an I-pair (A, A_0) such that*

(i) $A \cap N^v = \emptyset, A_0 \cap N = \emptyset$

(ii) f maps the triple (N_i, N_i^-, N_i^+) into the triple (A, A_0^-, A_0^+).

(iii) $i : (A, A_0) \to (X \setminus N^v, X \setminus N)$ maps $[\kappa_{0,1}]$ onto $[\mu_0^c] = [\mu_1^c] \in \Gamma(X \setminus N^v, X \setminus N)$

Then N is an isolating neighborhood for f and there exists a continuous surjection $\varphi : (\mathrm{Inv}\, N, f) \to (\Sigma_2, \sigma)$. Moreover, for any periodic sequence $\alpha \in \Sigma_2$ there exists an $x \in N$ such that $\varphi(x) = \alpha$ and x is periodic with the same proncipal period as α.

Note that in the above theorem the only assumption concerning f is the inclusion condition on the image of N_i, N_i^-, N_i^+ under f and such a condition is obviously inheritable. Thus in order to apply the above theorem it is enough to check this condition for a multivalued representation of f which is sufficiently close to f.

4. A few examples.

The following three theorems were obtained in computer assisted proofs based on the described geometric principle.

Theorem 4.1. *(see [21]) Consider the Hénon map $H_{ab} : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$H_{ab}(x, y) := (1 + y - ax^2, bx).$$

Then there exists a compact set $N \subset \mathbb{R}^2$ such that for (a, b) sufficiently close to $(1.4, 0.3)$ there exists a continuous surjection $\rho : (\mathrm{Inv}\, N, H_{ab}^7) \to (\Sigma_2, \sigma)$. Moreover, for any periodic sequence $\alpha \in \Sigma_2$ there exists an $x \in \mathrm{Inv}\, N$ such that $\rho(x) = \alpha$ and x is periodic with the same principial period as α.

The following theorem follows from Theorem 3.1 and from the computations performed in [11, 12].

Theorem 4.2. *Consider the Lorenz equations*

$$\begin{aligned} \dot{x} &= s(y-x) \\ \dot{y} &= Rx - y - xz \\ \dot{z} &= xy - qz, \end{aligned} \tag{1}$$

and the plane

$$P := \{(x,y,z) \mid z = 53\}.$$

For all parameter values (s,R,q) *sufficiently close to* $(45,54,10)$*, there exists a Poincare section* $N \subset P$ *such that the Poincare map* g *induced by (1) is Lipschitz and well defined. Furthermore, there exists a continuous surjection* $\rho : (\operatorname{Inv} N, g) \to (\Sigma_2, \sigma)$*. Moreover, for any periodic sequence* $\alpha \in \Sigma_2$ *there exists an* $x \in \operatorname{Inv} N$ *such that* $\rho(x) = \alpha$ *and* x *is periodic with the same principial period as* α*.*

Theorem 4.3. *(see* [22]*). Consider the Rössler equations*

$$\begin{aligned} \dot{x} &= -(y+z) \\ \dot{y} &= x + by \\ \dot{z} &= b + z(x-a), \end{aligned} \tag{2}$$

and the plane

$$P := \{(x,y,z) \mid x = 0\}.$$

For all parameter values in a sufficiently small neighbourhood of $(a,b) = (5.7, 0.2)$*, there exists a Poincare section* $N \subset P$ *such that the second iterate of the Poincaré map* g *induced by (2) is Lipschitz and well defined. Furthermore, there exists a continuous surjection* $\rho : (\operatorname{Inv} N, g^2) \to (\Sigma_2, \sigma)$*. Moreover, for any periodic sequence* $\alpha \in \Sigma_2$ *there exists an* $x \in \operatorname{Inv} N$ *such that* $\rho(x) = \alpha$ *and* x *is periodic with respect to* g^2 *with the same principial period as* α*.*

The computations necessary to establish the above results took respectively (approximately) 20sec, 90h, 44h CPU time of an IBM PC with a 486DX 50MHz processor.

5. Isolating neighbourhoods

Although the computations necessary to establish the chaotic behaviour in a Poincaré map by means of Theorem 3.1 are many times faster then those based on the existence of a transversal homoclinic point criterion (tens instead of thousands of hours of CPU time - compare [18]), they still are quite expensive in case of a Poincaré map. A solution wuold be to look for a chaos criterion based on the behaviour of the vector field only in order to avoid the expensive computation of the Poincaré map. In the following sections we introduce some machinery, which shows that such an approach is indeed possible.

Assume X is a locally compact metric space and $f : X \to X$ is a homeomorphism. Then f generates a *discrete dynamical system*

$$\pi_f : X \times \mathbb{Z} \ni (x, n) \to f(x, n) := f^n(x) \in X$$

denoted often simply by f.

Let $N \subset X$ be a compact set. The *invariant part* of N is given by

$$\operatorname{Inv} N := \{x \in N \mid \forall n \in \mathbb{Z}\ f^n(x) \in N\}$$

The set N is called an *isolating neighborhood* if any of the following two equivalent conditions is satisfied

$$\forall x \in \operatorname{bd} N\ \exists n \in \mathbb{Z}\ f(x, n) \notin N$$

$$\operatorname{Inv} N \subset \operatorname{int} N.$$

If N is an isolating neighborhood then the set $S := \operatorname{Inv} N$ is called an *isolated invariant set* . Replacing in the above definitions $\mathbb{Z}$ by $\mathbb{R}$ one obtains analogous concepts for flows.

Definition 5.1. *The pair $P = (P_1, P_2)$ of compact subsets of N will be called an index pair of S in N iff the following three conditions are satisfied*

$$x \in P_i,\ f(x) \in N \ \Rightarrow\ f(x) \in P_i, \quad i = 1, 2$$

$$x \in P_1,\ f(x) \notin N \ \Rightarrow\ x \in P_2$$

$$\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2).$$

Theorem 5.2. *Every isolating neighbourhood admits at least one index pair P.*

It can be easily derived from the definition of the index pair that f induces a map of pairs

$$f_P : (P_1, P_2) \ni x \to f(x) \in (P_1 \cup f(P_2), P_2 \cup f(P_2))$$

and the inclusion

$$i_P : (P_1, P_2) \ni x \to x \in (P_1 \cup (P_2), P_2 \cup f(P_2)).$$

induces an isomorphism in Alexander-Spanier cohomology.

Definition 5.3. *The endomorphism*

$$H^*(f_P) \circ H^*(i_P)^{-1} \ \textit{of}\ H^*(P),$$

where H^* denotes the Alexander-Spanier cohomology, will be called the index map associated with the index pair P and denoted by I_P.

6. The Conley index

Let $\mathcal{E}$ be a category. We consider the *category of endomorphisms* of $\mathcal{E}$ denoted by $\mathrm{Endo}(\mathcal{E})$ and defined as follows. The objects of $\mathrm{Endo}(\mathcal{E})$ are pairs (A, a), where $A \in \mathcal{E}$ and $a \in \mathcal{E}(A, A)$ is a distinguished endomorphism of A. The set of morphisms from $(A, a) \in \mathcal{E}$ to $(B, b) \in \mathcal{E}$ is the subset of $\mathcal{E}(A, B)$ consisting of exactly those morphisms $\varphi \in \mathcal{E}(A, B)$ for which $b\varphi = \varphi a$. We write $\varphi : (A, a) \to (B, b)$ to denote that φ is a morphism from (A, a) to (B, b) in $\mathrm{Endo}(\mathcal{E})$. We define the category of automorphisms of $\mathcal{E}$ as the full subcategory of $\mathrm{Endo}(\mathcal{E})$ consisting of pairs $(A, a) \in \mathrm{Endo}(\mathcal{E})$ such that $a \in \mathcal{E}(A, A)$ is an automorphism, i.e., both an endomorphism and an isomorphism in $\mathcal{E}$. The category of automorphisms of $\mathcal{E}$ will be denoted by $\mathrm{Auto}(\mathcal{E})$. There is a functorial embedding

$$\mathcal{E} \ni A \to (A, \mathrm{id}_A \in \mathrm{Auto}(\mathcal{E}))\,,$$

$$\mathcal{E}(A, B) \ni \varphi \to \varphi \in \mathrm{Auto}(\mathcal{E})(A, B)\,,$$

hence we can consider the category $\mathcal{E}$ as a subcategory of $\mathrm{Auto}(\mathcal{E})$.

Definition 6.1. *We will say that $F : \mathcal{C} \to Auto(\mathcal{E})$ is* normal*, if for each $(A, a) \in \mathcal{C}$ the morphism $F(a)$ is equal to the automorphism distinguished in $F(A, a)$.*

Theorem 6.2. *(see* [13]*) Assume $\mathcal{E}$ admits inverse limits. Then the inverse limit functor which sends $(A, a) \in Endo(\mathcal{E})$ to the inverse limit of the sequence*

$$\ldots \xrightarrow{a} A \xrightarrow{a} A \xrightarrow{a} A$$

is a normal functor.

Let us denote this functor by L.

Theorem 6.3. *(see* [13]*) Assume $f : X \to X$ is a homeomorphism and N is an isolating neighborhood with respect to f. Then $L(H^*(P), I_P)$ does not depend on the particular index pair in N.*

We define the *Conley index* of the isolating neighborhood N as $\mathrm{Con}^*(N) := L(H^*(P), I_P)$, where P is an index pair in N. Let us emphasize that the Conley index has the form of a pair

$$\mathrm{Con}^*(N) = (CH^*(N), \chi^*(N)),$$

where $CH^*(N)$ is a graded vector space and $\chi^*(N) : CH^*(N) \to CH^*(N)$ is a graded automorphism.

7. The Conley index for multivalued maps

Assume $F : X \times \mathbb{Z} \to X$ is an usc compact-valued map. We will say that F is a *multivalued discrete dynamical systeem* (mvdds) on x if the following conditons are satisfied

$$F(x, 0) = x$$

$$y \in F(x, n) \Leftrightarrow x \in F(y, -n)$$

$$nm \geq 0 \Rightarrow F(F(x, n), m) = F(x, n + m)$$

One can easily observe that if X is compact then any usc mv map $F : X \to X$ induces a mvdds.

Assume $N \subset X$. The set $\operatorname{Inv}(N, F)$ is defined as the set of $x \in N$ such that there exists a function $\sigma : \mathbb{Z} \to N$ satisfying $\sigma(0) = x$ and $\sigma(n + 1) \in F(\sigma(n))$.

The set N is said to be a pseudo isolating neighbourhood for F if

$$\forall x \in \operatorname{bd} N \; \exists n \in \mathbb{Z} \; F(x, n) \cap N = \emptyset$$

The set N is said to be an isolating neighbourhood for F if

$$B_{\operatorname{diam}_N F}(\operatorname{Inv} N) \subset \operatorname{int} N,$$

where $\operatorname{diam}_N F$ is the maximal diameter of the values of F in N.

Proposition 7.1. *(see* [14]*) Both the pseudo isolating neighborhood and the isolating neighborhood are inheritable properties. The isolating neighborhood is easily seen to be also strongly inheritable.*

Let N be an isolating neighbourhood for F. A pair $P = (P_1, P_2)$ of compact subsets $P_2 \subset P_1 \subset N$ is called an *index pair* if the following conditions are satisfied:

$$F(P_i) \cap N \subset P_i, \; i = 1, 2;$$

$$F(P_1 \backslash P_2) \subset N;$$

$$\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2)$$

Theorem 7.2. *(see* [9]*) Every isolating neighborhood admits an index pair.*

Proposition 7.3. *If P is an index pair for N then the inclusion*

$$i_P : F(P) \subset T(P) := (P_1 \cup F(P_2), P_2 \cup F(P_2))$$

induces an isomorphisms in the Alexander-Spanier cohomology.

In order to define the index map in the multivalued case we need mv maps which induce maps in cohomology. For this purpose we restrict our attention to mvdds such that F is given by a morphism in the sense of Górniewicz & Granas (comp. [5]).

Let $F_{P,T(P)} : P \to 2^{T(P)}$ be the restriction of F to domain P and codomain $T(P)$. The endomorphism $I_P := H^*\left(F_{P,T(P)}\right) \circ H^*(i_P)^{-1}$ of $H^*(P)$ is called the *index map* associated with the index pair P.

Theorem 7.4. *(see* [9]*) Let N be an isolating neighbourhood. Then $(L(H^*(P), I_P)$ is independent of the choice of an index pair P in N.*

As in the discrete case we define the *Conley index* of the isolating neighborhood N as $\mathrm{Con}^*(N) := L(H^*(P), I_P)$, where P is an index pair in N.

Theorem 7.5. *(see* [14]*) The Conley index is a strongly inheritable property.*

We say that N is an *isolating block* if

$$B_{\mathrm{diam}_N F}(F^{*-1}(N) \cap N \cap F(N)) \subset \mathrm{int}\, N$$

Theorem 7.6. *(see* [14]*) Assume F is usc and N is an isolating block for F. Define*

$$N^- := N \backslash F^{-1}(\mathrm{int}\, N).$$

Then (N, N^-) is an index pair for F in N.

Theorem 7.7. *(see* [14]*) Assume F is lsc and N is an isolating block for F. Define*

$$N^- := N \backslash \mathrm{int}(N \cap F^{-1}(N)).$$

Then (N, N^-) is an index pair for F in N.

8. The Conley index and chaos

The following criterion is the earliest topological criterion for chaos.

Theorem 8.1. *(see* [10]*) Assume that $N = N_0 \cup N_1$ is an isolating neighbourhood under f where N_0 and N_1 are disjoint compact sets and for $k = 0, 1$*

$$\mathrm{Con}^n(N_k) = \begin{cases} (Q, \mathrm{Id}) & if\, n{=}1 \\ 0 & otherwise. \end{cases}$$

Then $N_{kl} := (N_k \cap f(N_k)) \cup (N_k \cap f(N_l)) \cup (N_l \cap f(N_l))$ for $k, l \in \{0, 1\}, k \neq l$, are isolating neighbourhoods and if additionally $\chi^*(N_{kl})$ is not conjugate to identity then there exists a $d \in N$ and a continuous surjection $\rho : (\mathrm{Inv}\, N, f) \to (\Sigma_2, \sigma)$.

The followinng theorem was conjectured in [10] and proved by Szymczak in [20].

Theorem 8.2. *Assume X is a compact ANR and $N = N_0 \cup N_1 \subset X$ is an isolating neighbourhood under f where N_0 and N_1 are disjoint compact sets and the Conley index of N_0 is as in the previous theorem, whereas $\mathrm{Con}(N) = 0$. Then there exists a $d \in \mathbb{N}$ and a continuous surjection $\rho : (\mathrm{Inv}\, N, f) \to (\Sigma_2, \sigma)$. Moreover, for any periodic sequence $\alpha \in \Sigma_2$ there exists an $x \in \mathrm{Inv}\, N$ such that $\rho(x) = \alpha$ and x is periodic with respect to g^d with the same principial period as α.*

The above two theorems are formally weaker then Theorem 3.1 but their potential strength in case of a Poincaré map lies in the possibility of regaining the index information just from the vector field. We want to explain these ideas now.

If $\pi : X \times \mathbb{R} \to X$ is a flow and N is an isolating neighborhood with respect to π then the Conley index of $\pi_t : X \ni x \to \pi(x,t) \in X$ does not depend on t for $t > 0$. Thus the Conley index for flows is well defined. Numerical computations of such an index are much less expensive than the computations of the Conley index of a Poincaré map, because one needs to follow the flow only for a short period of time. If N admits a cross-section Ξ then there is a well defined Poincaré map $\Pi : U \to \Xi$, for some open $U \subset \Xi$. Unfortunately it turns out that it is impossible to regain the Conley index of the Poincare map from the Conley index for flows.

Let S^1 denote the unit circle in $\mathbb{R}^2$ with a fixed orientation.

Assume N is an isolating neighborhood with respect to a flow π admitting a cross-section Ξ. For each $x \in N$ denote by $r(x)$ the time of first return of the trajectory of x to Ξ. W.l.o.g. we may assume that $r(x) = 1$ for $x \in N \cap \Xi$. Thus r defines a surjective continuous map $\Gamma : N \to S^1$.

Let $P = (P_1, P_2)$ be an index pair in N. We will say that the index pair P is regular if the map

$$\omega_P : P_1 \ni x \to \sup\{t > 0 | \pi(x,[0,t]) \subset P_1 \backslash P_2\} \in \mathbb{R} \tag{3}$$

is continuous. One can show that for any open neighborhood U of S there exists a regular index pair in N.

Let $\Gamma(P)$ denote the disjoint union $P_1 \cup S^1$ divided by the following equivalence relation

$$x \approx y \text{ iff } x = y \text{ or } (x \in P_1 \text{ and } y = \Gamma(x)) \tag{4}$$

Notice that S^1 may be considered as a subsapce of $\Gamma(P)$.

Theorem 8.3. *(see [15]) The cohomology $H^*(\Gamma(P), S^1)$ does not depend of the choice of a regular index pair, i.e. it is an invariant of S.*

We will call this cohomology the Conley index of S over S^1.

Assume $\Delta \subset S^1$ is a closed interval. The fixed orientation of S^1 induces an order on Δ.

We will say that (X,p) is a trivial semibundle over Δ if $p : X \to \Delta$ is a continuous map such that for some $a \in \Delta$ there exists a continuous surjection $h : p^{-1}(a)\times\Delta \to X$ satisfying the following two properties

(i) $p(h(x,t)) = t$ for any $t \in \Delta, x \in p^{-1}(a)$

(ii) $h(x,t) = h(y,t), s > t$ imply $h(x,s) = h(y,s)$

We will say that (X,p) is a semibundle over S^1 if for any $a \in S^1$ there exists a closed interval Δ such that $a \in \operatorname{int}\Delta$ and $(p^{-1}(\Delta), p|_{p^{-1}(\Delta)})$ is a trivial semibundle over Δ.

Theorem 8.4. *(see [15]) Assume (X,p) is a semibundle over S^1. Then there is a collection of maps $\{p_{s,t} : p^{-1}([t]) \to p^{-1}([s]),\ s,t \in \mathbb{R}, s \le t\}$, unique up to a homotopy, such that*

$$p_{s+r,t+r} = p_{s,t} \tag{5}$$

$$p_{t,t} = \mathrm{id}_{p^{-1}(t)} \tag{6}$$

$$p_{r,s}p_{s,t} = p_{r,t} \tag{7}$$

The map $p_{t,t+1}$ will be called the monodromy map associated with the leave $p^{-1}([t])$.

Theorem 8.5. *(see [15]) Let P be an index pair in N. Then $\Gamma(P)$ is a semibundle over S^1. Let m be a monodromy map. Then $L(H^*(\Gamma(P),S^1),m)$ coincides with the discrete Conley index of the Poincaré map associated with S.*

The above theorem shows that the knowledge of the Conley index over S^1 allows one to reconstruct the Conley index of the Poincaré map. The numerical computation of the Conley index over S^1, like the computation of the classical Conley index in the flow sense, is much less expensive than the computation of the Conley index of a Poincaré map. Thus one can expect that the two presented Theorems characterizing chaos for discrete dynamical systems will become very useful in the rigorous numerical verification of chaos for flows admitting cross-section. Research in this direction is in progress.

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