ALGORITHMS AND THE CONLEY INDEX THEORY

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In the proof of chaos in the Lorenz equations [3] a computer was used to verify the assumptions of a chaos criterion presented in [2]. This required an algorithmic computation of Conley indexes of four appriopriately chosen isolating neighborhoods. The approach used in [3] was based on an ad hoc algorithm, designed to work just for the particular dynamical system. In order to make the technique introduced in [3] broadly applicable to various problems, a more systematic studies of algorithms computing the Conley index are needed. In this brief paper we present some ideas which lead in that direction. The detailed proofs will be published in [5].

The first step is to introduce a class of sets closed under elementary set theoretical and topological operations. For $\mathcal{A} \subset \mathcal{P}(X)$ and $A \subset X$ put $\operatorname{diam} \mathcal{A} := \sup\{ \operatorname{diam} a \mid a \in \mathcal{A} \}$, $\mathcal{A}(A) := \{ a \in \mathcal{A} \mid a \cap A \neq \emptyset \}$, $\mathcal{A}(x) := \mathcal{A}(\{ x \})$, $|\mathcal{A}| := \bigcup \mathcal{A}$. For $\mathcal{C} \subset \mathcal{A}$ put $\langle \mathcal{C} \rangle := \langle \mathcal{C} \rangle_{\mathcal{A}} := \{ x \in X \mid \mathcal{A}(x) = \mathcal{C} \}$. The family $\mathcal{A}$ will be called *a grid* in X if the elements of $\mathcal{A}$ are compact, non-empty subsets of X such that any non-empty compact subset of X intersects at least one and at most a finite number of elements of $\mathcal{A}$ and $\forall \mathcal{C} \subset \mathcal{A}\ \operatorname{cl} \langle \mathcal{C} \rangle = \bigcap \mathcal{C}$. By an *elementary representable set* over $\mathcal{A}$ we mean a set of the form $\langle \mathcal{C} \rangle$, where $\mathcal{C}$ is a finite subfamily of $\mathcal{A}$. The set A will be called *representable* if it is a finite union of elementary representable sets. It will be called *strongly representable* over $\mathcal{A}$, if it is a finite sum of a subfamily of $\mathcal{A}$. We will denote the family of elementary representable sets (representable sets, strongly representable sets) over $\mathcal{A}$ respectively by $\mathrm{ER}(\mathcal{A}), \mathrm{R}(\mathcal{A}), \mathrm{SR}(\mathcal{A})$.

Theorem 0.1. *The family of representable sets is closed under the set theoretical union, intersection, difference as well as topological closure and interior.*

Let $\mathcal{G}$ be a fixed grid in X and let $A \subset X$. Let us define the following two sets $o_{\mathcal{G}}(A) := \bigcup \{ E \in \mathrm{ER}(\mathcal{G}) \mid \operatorname{cl} E \cap A \neq \emptyset \}$ and $n_{\mathcal{G}}(A) := \bigcup \{ \gamma \in \mathcal{G} \mid \gamma \cap A \neq \emptyset \}$.

Theorem 0.2. *The set $o_{\mathcal{G}}(A)$ is the smallest open representable set containing A, i.e. if $U \in \mathrm{R}(\mathcal{G})$ is an open subset of A then $U \supset o_{\mathcal{G}}(A)$.*

We will say that the family $\mathcal{G}$ is *centered* if

$$\{ \gamma_1, \dots, \gamma_n \} \subset \mathcal{G},\ \gamma_i \cap \gamma_j \neq \emptyset \ \Rightarrow\ \gamma_1 \cap \ldots \cap \gamma_n \neq \emptyset.$$

Assume that $\mathcal{G}, \mathcal{G}'$ are two centered grids in X. We will say that $\mathcal{G}'$ is a *subgrid* of $\mathcal{G}$ if the following two conditions are satisfied

$$\forall \tau \in \mathcal{G}' \exists \sigma \in \mathcal{G}\ \tau \subset \sigma \tag{1}$$

$$\forall \tau \in \mathcal{G}' \forall E_1, E_2 \in \mathrm{ER}(\mathcal{G}) \tau \cap \operatorname{cl} E_i \neq \emptyset \ \Rightarrow\ \tau \cap \operatorname{cl} E_1 \cap E_2 \neq \emptyset. \tag{2}$$

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We now turn our attention to multivalued dynamics. First we recall that for any set X the notation $\mathcal{P}X$ will stand for the family of subsets of X. By a *multivalued map* $F : X \rightrightarrows Y$ we mean a map $F : X \to \mathcal{P}Y$. The *domain* of a multivalued map $F : X \rightrightarrows Y$ is the set $\operatorname{dom} F := \{ x \in X \mid F(x) \neq \emptyset \}$ and the image of F is defined as the set $\operatorname{im} F := \bigcup_{x\in X} F(x)$. For $A \subset X, B \subset Y$ we define *the image, strong preimage* and *weak preimage* of A, B under F as the sets $F(A) := \bigcup_{x\in A} F(x)$, $F^{o-1}(B) := \{ x \in \operatorname{dom} F \mid F(x) \subset B \}$, $F^{*-1}(B) := \{ x \in X \mid F(x) \cap B \neq \emptyset \}$. The inverse of F is defined as a multivalued map $F^{-1} : Y \ni y \to \{ x \in X \mid y \in F(x) \} \subset X$. If $F : X \rightrightarrows Y$ and $G : Y \rightrightarrows Z$ are two multivalued maps then we define their composition as a multivalued map $G \circ F : X \ni x \to G(F(x)) \subset Z$.

Assume X, Y are two locally compact metric spaces with given grids $\mathcal{G}$ and $\mathcal{H}$. Let $f : X \multimap Y$ be a continuous map. We say that a multivalued representable map $F : X \rightrightarrows Y$ is a *representation of f over $\mathcal{G}, \mathcal{H}$* if $\operatorname{dom} f \subset \operatorname{dom} F$ and $x \in \operatorname{dom} f$ implies $f(x) \in F(x)$.

Theorem 0.3. *If $F : X \rightrightarrows Y$ is representable, $A \in \mathrm{R}(\mathcal{G})$, $A' \in \mathrm{R}(\mathcal{H})$ then* $\operatorname{dom} F$, $F^{o-1}(A')$, $F^{*-1}(A') \in \mathrm{R}(\mathcal{G})$ *and* $F(A) \in \mathrm{R}(\mathcal{H})$. *Moreover, F^{-1} is representable.*

It is convenient to introduce a combinatorial representation of f as a multivalued map $\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{H}$ such that $\# \operatorname{dom} \mathcal{F} < \infty$, for every $g \in \mathcal{G} \# \mathcal{F}(g) < \infty$, the domain of f is contained in $|\operatorname{dom} \mathcal{F}|$ and $x \in \operatorname{dom} f, g \in \operatorname{dom} \mathcal{F}, x \in g$ implies $f(x) \in |\mathcal{F}(g)|$. With a combinatorial multivalued map $\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{H}$ we associate the following two multivalued maps:

$$\lfloor \mathcal{F} \rfloor : |\mathcal{G}| \ni x \to \bigcap_{g\in\mathcal{G}(x)} |\mathcal{F}(G)| \tag{3}$$

$$\lceil \mathcal{F} \rceil : |\mathcal{G}| \ni x \to \bigcup_{g\in\mathcal{G}(x)} |\mathcal{F}(G)| \tag{4}$$

Theorem 0.4. *If $\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{H}$ is a combinatorial representation of f, then $\lfloor \mathcal{F} \rfloor$ and $\lceil \mathcal{F} \rceil$ are representations of f. Moreover, $\lfloor \mathcal{F} \rfloor$ is lower semicontinous and $\lceil \mathcal{F} \rceil$ is upper semicotinuous.*

The following theorem shows that every Lipschitz continuous map may be approximated by its multivalued representations.

Theorem 0.5. *Assume $\mathcal{G}_n, \mathcal{H}_n$ are sequences of grids respectively in X and Y such that* $\operatorname{diam} \mathcal{G}_n \to 0$, $\operatorname{diam} \mathcal{H}_n \to 0$. *Let $f : X \multimap Y$ be a Lipschitz function such that* $\operatorname{dom} f$ *is relatively compact. Then there exist sequences $F_n, G_n : X \rightrightarrows Y$ such that F_n, G_n are representations of f, F_n is lower semicontinuous and G_n is upper semicontinuous and $F_n \to f$, $G_n \to f$.*

The reader is referred to [1] and [4] for a detailed explanation what is the role of multivalued dynamics in reducing the computation of the Conley index to a combinatorial problem.

Let $N \subset X$ be a compact set and $F : X \rightrightarrows X$ be a representable map. Let $F_N : X \ni x \to N \cap F(x) \subset X$.

Proposition 0.6. *If F is representable and N is representable then F_N is representable.*

Define the positive and negative *invariant set* of N by

$$\mathrm{Inv}^+(N,F) := \{x \in N \mid \exists \sigma : \mathbb{Z}^+ \to N \text{s.t. } \sigma(0) = x \text{ and } \sigma(n+1) \in F(\sigma(n))\}.$$

and

$$\mathrm{Inv}^-(N,F) := \{x \in N \mid \exists \sigma : \mathbb{Z}^- \to N \text{s.t. } \sigma(0) = x \text{ and } \sigma(n+1) \in F(\sigma(n))\}.$$

Define the invariant part of N, denoted $\mathrm{Inv}\, N$, as the intersection of the positive and negative invariant parts. The set N is said to be an *isolating neighborhood* for F if $B_{\mathrm{diam}_N F}(\mathrm{Inv}\, N) \subset \mathrm{int}\, N$.

The computation of the Conley index is reduced to the computation of certain homology if one is able to construct for a given representable isolating neighborhood an representable *index pair*, i.e. a pair $P = (P_1, P_2)$ of compact representable subsets $P_2 \subset P_1 \subset N$ such that $F(P_i) \cap N \subset P_i$, for $i = 1, 2$, $F(P_1 \backslash P_2) \subset N$ and $\mathrm{Inv}\, N \subset \mathrm{int}(P_1 \backslash P_2)$. The following relusts show that such a computation is indeed possible.

Theorem 0.7. *Let N be a representable isolating neighborhood for a representable multivalued dynamical system generated by F. Then there exist numbers $p, q \in \mathbb{N}$ such that*

$$\mathrm{Inv}^-(N,F) = \bigcap_{i=0,\infty} F_N^i(N) = \bigcap_{i=0,p} F_N^i(N) \tag{5}$$

$$\mathrm{Inv}^+(N,F) = \bigcap_{i=0,\infty} F_N^{*-i}(N) = \bigcap_{i=0,q} F_N^{*-i}(N) \tag{6}$$

Corollary 0.8. *If N is representable and F is representable then $\mathrm{Inv}^- N, \mathrm{Inv}^+ N$ and $\mathrm{Inv}\, N$ are representable.*

Theorem 0.9. *Assume $F : X \rightrightarrows X$ is representable uppersemicontinuoous multivalued map and N is a representable isolating neighborhood with respect to a grid $\mathcal{G}$ in X. Let $\mathcal{H}$ be a subgrid of $\mathcal{G}$. Put $P_1 := n_{\mathcal{H}}(\mathrm{Inv}^- N)$, $P_2 := P_1 \setminus o_{\mathcal{H}}(\mathrm{Inv}^+ N)$. Then P_1, P_2 are representable with respect to $\mathcal{H}$ and $P := (P_1, P_2)$ is an index pair for F in N.*

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