CONLEY INDEX FOR DISCRETE MULTIVALUED DYNAMICAL SYSTEMS

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ABSTRACT

The definitions of isolating block, index pair, and the Conley index, together with the proof of homotopy and additivity properties of the index are generalized for discrete multivalued dynamical systems. That generalization provides a theoretical background of numerical computation used by Mischaikow and Mrozek in their computer assisted proof of chaos in the Lorenz equations, where finitely represented multivalued mappings appear as a tool for discretisation.

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1. Introduction

The aim of this paper is to construct the Conley index theory for discrete multivalued dynamical systems, i.e. for iterates of multivalued maps. The Conley index is a topological invariant defined for isolated invariant sets in the theory of dynamical systems. Its original construction by C. Conley and his students (comp. [3]) concerned flows on locally compact metric spaces. Later the theory was generalized to arbitrary metric spaces [16, 2], to multivalued flows [12], and to discrete dynamical systems [15, 13, 4]. The Conley index theory is similar in spirit to the fixed point index theory (comp. [5]) but, at least potentially, it has much broader applications. Apart from stationary (fixed) points, the theory provides existence theorems concerning bounded trajectories, heteroclinic connections, and recently also periodic trajectories [9] and chaos [10]. The main drawback of the theory is the fact that the analysis necessary to check assumptions of such theorems in concrete examples is often too complicated to be successfully preformed, despite the fact that numerical experiments indicate that the assumptions are satisfied. This leads to the concept of computer assisted rigorous verification of such assumptions [14]. Recently the first complete and rigorous result on chaos in the Lorenz equations [11] was obtained this way on the basis of the ideas of the presented paper.

It turns out that the theory of multivalued maps can be fruitfully used in studying single valued maps via computer assisted rigorous proofs. This is in contrast with the traditional motivation for studying multivalued maps, which comes from some problems in the control theory and the theory of differential equations without uniqueness. Multivalued maps arise in these theories quite naturally as the

t-translations generated by the associated multivalued flow. The most often used technique in the research on multivalued maps consists in carrying over theorems from the single valued case by means of a sequence of single valued maps approximating the multivalued map of interest. Thus, the theory of multivalued maps in this setting seems not to be a useful tool in studying single valued problems.

The situation is different when a single valued map is investigated numerically. By their very nature numerical computations of a map cannot give us exact results but, when combined with precise error estimates, they provide sets where the exact results are located (that kind of approach is a straightforward generalization of the interval arithmetic; comp. [1]). Thus we obtain a multivalued map such that the single valued map is its selector. If we want to get some existence results concerning the dynamics of the single valued map based on some topological invariants, it seems reasonable to try to obtain these invariants from the multivalued map. Thus what we need are some topological invariants, which can be computed for the multivalued map and applied to the single valued selector to get the required existence results. This explains our interest in the Conley index theory for discrete multivalued dynamical systems.

The main obstacle in the construction of the Conley index for discrete multivalued dynamical systems is the fact that, unlike the generalization to the case of multivalued flows or single valued discrete dynamical systems, the concept of an isolating neighborhood seems to be no longer useful when carried over directly from the classical case of a single valued flow. Let us recall that a compact set N is called an isolating neighborhood if the maximal invariant subset of N is in a positive distance from the boundary of N. The definition we propose for discrete multivalued dynamical systems requires that this distance is greater than the maximal diameter of the values of a multivalued map F of interest. From the point of view of a single valued selector f of F this condition means that N is an isolating neighborhood for f and all its perturbations in the bounds indicated by the multivalued map F.

Since the Conley index is based on cohomology (or homotopy) theory, the class of considered multivalued maps must induce maps at least in cohomology. A class of such multivalued maps, called admissible mv maps (a subclass of upper semicontinuous maps) was developed in [6, 7, 8] for the sake of the fixed point theory and we use that class in our approach. It would be interesting to answer the question what other classes of multivalued maps would be suitable for a similar construction.

Single valued continuous maps constitute a very special subclass of admissible mv maps. The theory developed in this paper, when applied to single valued continuous maps reduces in a trivial way to the Conley index theory for maps as developed in [13].

The organization of the paper is as follows. In the second section we construct and study index pairs. In the following section the Conley index is constructed. Homotopy and additivity properties are proved in the third section. In the last section some elementary examples are discussed.

2. Construction and properties of index pairs

Let us recall that a mapping $F : X \to \mathcal{P}(Y)$, where X, Y are metric spaces and $\mathcal{P}(Y)$ is the set of all subsets of Y, is called *upper semicontinuous (usc)* if $F^{-1}(A) := \{x \in X : F(x) \cap A \neq \emptyset\}$ is closed for any closed $A \subset Y$ or, equivalently, if the set $\{x \in X : F(x) \subset U\}$ is open for any open $U \subset Y$. If $A \subset X$, we denote by $F(A)$ the union $\bigcup\{F(x) : x \in A\} \subset Y$ and not a subset of $\mathcal{P}(Y)$. Given a positive integer n, F^n denotes the $n'th$ superposition of F defined recursively by $F^n(x) := F(F^{n-1}(x))$. The *graph of* F is the set $G(F) := \{(x, y) \in X \times Y : y \in F(x)\}$. Let us recall that any usc mapping with compact values has a closed graph and it sends compact sets to compact sets. If $F : X \to \mathcal{P}(Y)$ is usc then the set $D(F) := \{x \in X : F(x) \neq \emptyset\}$ (called the *effective domain of* F) is closed.

Let now (X, d) be a given locally compact metric space. If $A \subset X$, we denote the boundary of A by $\operatorname{bd} A$, its interior by $\operatorname{int} A$, and we let $B_\varepsilon(A) := \{x \in X : d(x, A) < \varepsilon\}, \varepsilon > 0$. We denote the sets of all integers, non negative intergers, and nonpositive integers by $\mathbb{Z}, \mathbb{Z}^+$, and $\mathbb{Z}^-$, respectively. By an *interval* we mean an interval in $\mathbb{Z}$, i.e. an intersection of a closed real interval with $\mathbb{Z}$.

Definition 2.1. An usc mapping $F : X \times \mathbb{Z} \to \mathcal{P}(X)$ with compact values is called a *discrete multivalued dynamical system (dmds)* if the following conditions are satisfied:

(i) For all $x \in X, \quad F(x, 0) = \{x\}$;
(ii) For all $n, m \in \mathbb{Z}$ with $nm \geq 0$ and all $x \in X, \quad F(F(x, n), m) = F(x, n + m)$;
(iii) For all $x, y \in X, \quad y \in F(x, -1) \Longleftrightarrow x \in F(y, 1)$.

An usc mapping $F : X \times \mathbb{Z}^+ \to \mathcal{P}(X)$ is called a discrete multivalued semidynamical system (dmss) if it satisfies conditions (i) and (ii) above.

We use the notation $F^n(x) := F(x, n)$. Note that F^n coincides with a superposition of $F^1 : X \to \mathcal{P}(X)$ or its inverse $(F^1)^{-1}$. This justifies that we will call F^1 the *generator* of the dmds F. We will usually denote the generator simply by F and identify it with the dmds. This will cause no misunderstanding unless a value of F is considered but in that case the meaning will be clear from the number of arguments.

If X is compact then, unlike the single valued case, every dmss F can be extended to a dmds, as the following proposition shows.

Proposition 2.2. Assume $F : X \times \mathbb{Z}^+ \to \mathcal{P}(X)$ is a dmss on a compact set X. Then $\tilde{F} : X \times \mathbb{Z} \to \mathcal{P}(X)$ defined by

$$\tilde{F}(x, n) := \begin{cases} F(x, n) & \text{for } n \in \mathbb{Z}^+ \\ \{y \in X : x \in F(y, -n)\} & \text{for } n \in \mathbb{Z}^- \end{cases}$$

is a dmds on X such that $\tilde{F}|_{X \times \mathbb{Z}^+} = F$.

Proof. Conditions (i)-(iii) are obviously satisfied. The closed graph property of F implies that $\tilde{F}$ is usc and has compact values.

The above proposition shows that, at least in case of a compact X, it is sufficient to study only discrete multivalued dynamical systems. In all what follows, we assume that F is a given dmds.

Definition 2.3. Let I be an interval in $\mathbb{Z}$ with $0 \in I$. A single valued mapping $\sigma : I \to X$ is a *solution for F through $x \in X$* if $\sigma(n+1) \in F(\sigma(n))$ for all $n, n+1 \in I$, and $\sigma(0) = x$.

Note that if $\sigma : I \to X$ is a solution for F then $\sigma(n) \in F^n(\sigma(0))$ for all $n \in I$ (The proof is straightforward by induction on m and k, where $I = [-k, m], k, m \in \mathbb{Z}^+$). The existence of a solution through x forces $F^n(x)$ to be nonempty for $n \in I$.

Given a subset $N \subset X$, we introduce the following notation:

$$\begin{aligned}
\operatorname{inv}^+ N :&= \{x \in N : \text{there exists a solution } \sigma : \mathbb{Z}^+ \to N \text{ for } F \text{ through } x\}\\
\operatorname{inv}^- N :&= \{x \in N : \text{there exists a solution } \sigma : \mathbb{Z}^- \to N \text{ for } F \text{ through } x\}\\
\operatorname{inv} N :&= \{x \in N : \text{there exists a solution } \sigma : \mathbb{Z} \to N \text{ for } F \text{ through } x\}
\end{aligned}$$

By (i) we have : $\operatorname{inv} N = \operatorname{inv}^+ N \cap \operatorname{inv}^- N$.

Let $\operatorname{diam}_N F := \sup\{\operatorname{diam} F(x) : x \in N\}$ and $\operatorname{dist}(A, B) := \min\{d(x, y) : x \in A, y \in B\}$, $A, B \subset X$.

Definition 2.4. A compact subset $N \subset X$ is called

(a) an *isolating neighbourhood for* F if

$$B_{\operatorname{diam}_N F}(\operatorname{inv} N) \subset \operatorname{int} N \tag{2.1}$$

or equivalently

$$\operatorname{dist}(\operatorname{inv} N, \operatorname{bd} N) > \operatorname{diam}_N F$$

(b) an *isolating block for* F if

$$B_{\operatorname{diam}_N F}(F^{-1}(N) \cap N \cap F(N)) \subset \operatorname{int} N \tag{2.2}$$

or equivalently

$$\operatorname{dist}(F^{-1}(N) \cap N \cap F(N), \operatorname{bd} N) > \operatorname{diam}_N F$$

A straightforward verification shows that (2.2) implies (2.1), i.e. every isolating block is an isolating neighbourhood but not necessarily vice versa. The importance of the notion of isolating block lies in the fact that it may be verified even if the set $\operatorname{inv} N$ is not known, which is usually the case.

Notice that when F is single valued then $\operatorname{diam}_N F = 0$ and conditions (2.1), (2.2) reduce to standard definitions of the isolating neighbourhood and isolating block.

Definition 2.5. Let N be an isolating neighbourhood for F. A pair $P = (P_1, P_2)$ of compact subsets $P_2 \subset P_1 \subset N$ is called an *index pair* if the following conditions are satisfied:

(a) $F(P_i) \cap N \subset P_i$, $i = 1, 2$;
(b) $F(P_1 \backslash P_2) \subset N$;
(c) $\operatorname{inv} N \subset \operatorname{int}(P_1 \backslash P_2)$

Our first aim is to prove the following result:

Theorem 2.6. *Let N be an isolating neighbourhood for F and W a neighbourhood of* $\operatorname{inv} N$. *Then there exists an index pair P for N with $P_1 \backslash P_2 \subset W$.*

The proof is based on several lemmas. First, given $N \subset X$, $x \in N$, and $n \in \mathbb{Z}^+$, the following notation will be used:

$$F_{N,n}(x) := \{y \in N : \quad \text{there exists a solution} \quad \sigma : [0,n] \to N \quad \text{for } F \quad \text{such that} \\ \sigma(0) = x \quad \text{and} \quad \sigma(n) = y\}; \tag{2.3}$$

$$F_{N,-n}(x) := \{y \in N : \quad \text{there exists a solution} \quad \sigma : [-n,0] \to N \text{ for } F \text{ such that} \\ \sigma(-n) = y \quad \text{and} \quad \sigma(0) = x\}; \tag{2.4}$$

$$F_N^+(x) := \bigcup_{n \in \mathbb{Z}^+} F_{N,n}(x); \; F_N^-(x) := \bigcup_{n \in \mathbb{Z}^+} F_{N,-n}(x). \tag{2.5}$$

Proposition 2.7. *If $N \subset X$ is compact, then $F_{N,n} : N \to \mathcal{P}(N)$ is usc for any $n \in \mathbb{Z}$.*

Proof. It is enough to prove the assertion for $n \in \mathbb{Z}^+$ since the case of a negative n is analogous. Suppose that $F_{N,n}$ is not usc. Then there exists an open subset U of N, a point $x \in N$ with $F_{N,n}(x) \subset U$ and a convergent sequence $x_k \to x$, $\{x_k\} \subset N$ with $F_{N,n}(x_k) \cap (N - U) \neq \emptyset$. Consequently for each k, there exists a solution $\sigma_k : [0,n] \to N$ for F through x_k such that $\sigma_k(m_k) \in N - U$ for some $m_k \in [0,n]$. Passing to a subsequence we may assume that $m_k \equiv m \in [0,n]$ for all k. Since $N \backslash U$ is compact, we may assume that $\sigma_k(i) \to y_i \in N$ for $i \in [0,n]$. Then $y_m \in N \backslash U$. We define $\sigma(i) = y_i, I \in [0,n]$. By the closed graph property of F, $\sigma(i+1) \in F(\sigma(i))$, moreover $\sigma(0) = x$ and $\sigma(m) \in N \backslash U$. That contradicts the hypothesis $F_{N,n}(x) \subset U$. $\square$

Lemma 2.8. *Let $N \subset X$ be compact and suppose that $D(F_{N,n}) \neq \emptyset$ for all $n \in \mathbb{Z}^+$. Then* $\operatorname{inv} N \neq \emptyset$. *Moreover* $\operatorname{inv}^{(\pm)} N = \bigcap \{D(F_{N,n}) : n \in \mathbb{Z}$ *(respectively* $n \in \mathbb{Z}^+, n \in \mathbb{Z}^-)\}$.

Proof. It is easy to see that $\{D(F_{N,n})\}_{n=0,1,2,\ldots}$ is a decreasing sequence of compact sets, therefore its intersection K is nonempty. We prove that $\operatorname{inv}^+ N = K$. The proof for $\operatorname{inv}^- N$ is analogous and the conclusion for $\operatorname{inv} N$ follows from (2.1). The inclusion $\operatorname{inv}^+ N \subset K$ is obvious. Suppose that $x \in K$. Then, for each $n \in \mathbb{Z}^+$ there exists a solution $\sigma_n : [0,n] \to N$ for F through x. We construct a solution $\sigma : \mathbb{Z}^+ \to N$ for F through x by induction. Evidently, $\sigma(0) = x$. Suppose that $\sigma_{|[0,n]}$ is constructed and there is a sequence $\{k_i\}$ with $k_i > n$ such that $\sigma_{k_i} \to \sigma(n)$ as $i \to \infty$. Since N is compact, passing again to a subsequence we may assume that $\sigma_{k_i}(n+1) \to y_{n+1} \in N$ as $i \to \infty$. By the closed graph property, $y_{n+1} \in F(\sigma(n))$, so it remains to put $\sigma(n+1) := y_{n+1}$. $\square$

Lemma 2.9. *Let $N \subset X$ be compact. Then*

(a) *The sets* $\operatorname{inv}^+ N$, $\operatorname{inv}^- N$, *and* $\operatorname{inv} N$ *are compact;*
(b) *If* A *is compact with* $\operatorname{inv}^- N \subset A \subset N$ *then* $F_N^+(A)$ *is compact.*

Proof. (a) Since N is compact and $F_{N,n}$ is usc, the set $D(F_{N,n}^{(\pm)})$ is compact for any given n. The intersection of a family of compact sets is compact, hence the conclusion.
(b) It is sufficient to show that $F_N^+(A)$ is closed. Let $\{y_k\}$ be a sequence of points in $N, \sigma_k : [0, n_k] \to N$ a solution for F with $\sigma(0) \in A$ and $\sigma_k(n_k) = y_k$, for each k. Let $y_k \to y \in N$. We need to show that $y \in F_N^+(A)$.

Case 1. $\{n_k\}$ is bounded: then, passing to a subsequence if necessary, we may assume that $n_k \equiv n$ for all k. Since N and A are compact, we may assume that $\sigma_k(i) \to y_i \in N,\ i = 0, 1, \dots n,\ y_0 \in A$, and $y_n = y$.

By the closed graph property, $y_{i+1} \in F(y_i),\ i, i+1 \in [0, n]$, one may therefore define $\sigma(i) := y_i, i = 0, 1, \dots n$. It shows that $y \in F_{N,n}(A)$.

Case 2. $\{n_k\}$ is unbounded: then, passing to a subsequence, we may assume that $\{m_k\}$ is increasing and, by restricting the interval $[0, n_k]$, that $n_k = k$. Let $\sigma_k'(i) = \sigma_k(i + k),\ i \in [-k, 0]$. Then $\sigma_k' : [-k, 0] \to N$ is a solution for F with $\sigma_k'(-k) \in N$ and $\sigma_k'(0) = y_k$. Then $y_k \in D(F_{N,-k})$ and, by the same argument as in the proof of Lemma 2.8, $y \in \operatorname{inv}^- N \subset A$. □

Lemma 2.10. *Let* $K \subset N$ *be compact subsets of* X *such that* $K \cap \operatorname{inv}^+ N = \emptyset$ *(respectively* $K \cap \operatorname{inv}^- N = \emptyset$*). Then*

(a) $F_{N,n}(K) = \emptyset$ *for all but finitely many* $n > 0$ *(respectively* $n < 0$*);*
(b) *The mapping* F_N^+ *(respectively* F_N^-*) is usc on* K*;*
(c) $F_N^+(K) \cap \operatorname{inv}^+ N = \emptyset$ *(respectively* $F_N^-(K) \cap \operatorname{inv}^- N = \emptyset$*).*

Proof. (a) Assume that $K \cap \operatorname{inv}^+ N = \emptyset$. By Lemma 2.8, for each $x \in K$ there exists n_x such that $F_{N,n_x}(x) = \emptyset$. Since F_{N,n_x} is usc, there exists V_x such that $F_{N,n_x}(V_x) = \emptyset$. Let $\{V_{x_1}, \dots V_{x_k}\}$ be a finite covering of K. Now if $m \geq \max\{n_{x_i} : i = 1, 2, \dots k\}$, then $F_{n,m}(K) = \emptyset$. The proof for F_N^- is analogous.
(b) Follows from (a) and Proposition 2.7 since the union of finitely many usc maps is usc.
(c) is straightforward □

Lemma 2.11. *Let* $N \subset X$ *be compact. Then for any neighbourhood* V *of* $\operatorname{inv}^- N$ *there exists a compact neighbourhood* A *of* $\operatorname{inv}^- N$ *such that* $F_N^+(A) \subset V$.

Proof. By Lemma 2.10 there exists $m \in \mathbb{Z}^+$ such that

$$F_{N,-m}(N \backslash V) = \emptyset \tag{2.6}$$

Since $F_{N,m}$ is usc, one can find for every $x \in \operatorname{inv}^- N$ a compact neighbourhood V_x of x such that $F_{N,m}(V_x) \subset V$. By the compactness of $\operatorname{inv}^- N$ one can select a finite covering $\{V_{x_1}, \dots V_{x_k}\}$ of $\operatorname{inv}^- N$. Let $A = \bigcup_{i=1}^{k} V_{x_i}$. Then A is a compact neighbourhood of $\operatorname{inv}^- N$ such that $F_{N,m}(A) \subset V$. It remains to show that $F_N^+(A) \subset V$. Indeed, let $y \in F_N^+(A)$. Then there exists $n > 0$ and $x \in A$ such that $y \in F_{N,n}(x)$. If $n \leq m$, we are already done. If $n > m$, we note that $x \in F_{N,-n}(y) \subset F_{N,-m}(y)$ and (2.6) implies that $y \in V$. $\square$

Proof of Theorem 2.6 Since $\operatorname{inv} N \subset \operatorname{int} N$, we may assume that $W \subset \operatorname{int} N$. We may also assume that $F(W) \subset \operatorname{int} N$. Indeed, let

$$0 < \varepsilon < \operatorname{dist}(\operatorname{inv} N, \operatorname{bd} N) - \operatorname{diam} F$$

and let $\gamma := \varepsilon + \operatorname{diam} F$. Then $B_\gamma(\operatorname{inv} N) \subset \operatorname{int} N$ and we may intersect W with the open set $\{x \in X : F(x) \subset B_\gamma(\operatorname{inv} N)\}$. Let U and V be open neighbourhoods of $\operatorname{inv}^+ N$ and $\operatorname{inv}^- N$, respectively, such that $U \cap V \subset W$. Let A be given for V by Lemma 2.11. We define

$$P_1 = F_N^+(A),\ P_2 = F_N^+(P_1 \backslash U). \tag{2.7}$$

Then $P_1 \subset V$ and $P_1 \backslash U \subset P_2$ which implies that $P_1 \backslash P_2 \subset U$. Therefore $P_1 \backslash P_2 \subset U \cap V \subset W$. We verify that (P_1, P_2) is an index pair. P_1 is compact by Lemma 2.9 and P_2 is compact by Lemma 2.10 (b), since $P_1 \backslash U$ is compact. Next, $P_2 \subset F_N^+(P_1) \subset P_1$. To verify (a), let $x \in P_i$ and $y \in F(x) \cap N$. Then there exists a solution $\sigma : [0, n] \to N$ with $\sigma(n) = x$ and $\sigma(0) \in A$ in the case $i = 1$, $\sigma(0) \in P_1 \backslash U$ in the case $i = 2$, so one may extend σ to $[0, n+1]$ by puting $\sigma(n+1) = y$. Hence $y \in P_i$. Since $P_1 \backslash P_2 \subset W$ and $W \subset \operatorname{int} N$, (b) is verified. In order to verity (c), observe that P_1 is a neighbourhood of $\operatorname{inv}^- N$ and, by Lemma 2.10 (c), $N \backslash P_2$ is a neighbourhood of $\operatorname{inv}^+ N$. Therefore $P_1 \backslash P_2 = P_1 \cap (N \backslash P_2)$ is a neighbourhood of $\operatorname{inv}^- N \cap \operatorname{inv}^+ N = \operatorname{inv} N$. $\square$

We shall now discuss several properties of index pairs which will be used in the next section.

Proposition 2.12.

(i) *If P is an index pair for N, then $(P_1 \cup F(P_2)) \backslash (P_2 \cup F(P_2)) = P_1 \backslash P_2$*
(ii) *if P and Q are index pairs for N, then so is $P \cap Q$.*
(iii) *If $P \subseteq Q$ are index pairs for N, then so are $(P_1, P_1 \cap Q_2)$ and $(P_1 \cup Q_2, Q_2)$.*

Proof. (i) One verifies easily that $(P_1 \cup F(P_2)) \backslash (P_2 \cup F(P_2)) = P_1 \backslash P_2 \backslash F(P_2) \subset P_1 \backslash P_2$. To prove the inverse inclusion, let $x \in P_1 \backslash P_2$. Then $x \in M$ and if $x \in F(P_2)$, the property (a) of index pairs implies that $x \in F(P_2)$, a contradiction.
(ii) Veritication of (a) and (b) is obvious. For (c), let us note that $\operatorname{int}(P_1 \backslash P_2) \cap \operatorname{int}(Q_1 \backslash Q_2) \subset \operatorname{int}(P_1 \cap Q_1 \backslash (P_2 \cup Q_2)) \subset \operatorname{int}(P_1 \cap Q_1 \backslash P_2 \cap Q_2)$.

(iii) is a routine verification. $\square$

Lemma 2.13. *Let $P \subset Q$ be index pairs for N such that $P_1 = Q_1$ or $P_2 = Q_2$. Define a pair of sets $G(P,Q)$ by*

$$G_i(P,Q) = P_i \cup (F(Q_i) \cap N), \; i = 1,2. \tag{2.8}$$

Then (1) *If $P_i = Q_i$ then $G_i(P,Q) = P_i = Q_i$, $i = 1,2$;*
(2) *$P \subset G(P,Q) \subset Q$;*
(3) *$G(P,Q)$ is an index pair;*
(4) *$F(Q_i) \cap N \subset G_i(P,Q)$, $i = 1,2$.*

Proof. (4) is obvious and (1) immediate from the property (a) of index pairs. The first inclusion in (2) is obvious and the second is an immediate consequence of the first one and the property (a) satisfied by Q. It remains to prove (3). For (a), let $x \in G_i(P,Q)$ and $y \in F(x) \cap N$. If $x \in P_i$ then obviously $y \in G_i(P,Q)$. If $x \in F(Q_i) \cap N$ then $x \in Q_i$ hence $y \in F(Q_i) \cap N \subset G_i(P,Q)$. If $x \in F(Q_i) \cap N$ then $x \in Q_i$ hence $y \in F(Q_i) \cap N \subset G_i(P,Q)$. For (b) let us note that

$$G_1(P,Q) \backslash G_2(P,Q) \subset Q_1 \backslash G_2(P,Q) \subset Q_1 \backslash P_2$$

and either $Q_1 \backslash P_2 = Q_1 \backslash Q_2$, or $F(Q_1 \backslash P_2) \subset N$. For (c), let us note that $\operatorname{inv} N \subset \operatorname{int}(P_1 \backslash P_2) \cap \operatorname{int}(Q_1 \backslash Q_2)$ so (c) will follow if we verify that $(P_1 \backslash P_2) \cap (Q_1 \backslash Q_2) \subset G_1(P,Q) \backslash G_2(P,Q)$. Indeed, let $y \in (P_1 \backslash P_2) \cap (Q_1 \backslash Q_2)$. Then $y \in G_1(P,Q)$ and it remains to show that $y \in F(Q_2) \cap N$. For, if $y \in F(Q_2) \cap N$, then by (a) $y \in Q_2$, a contradiction. $\square$

Lemma 2.14. *Let $P \subseteq Q$ be index pairs such that $P_1 = Q_1$ or $P_2 = Q_2$. Then there exists a sequence of pairs*

$$P = Q^n \subset Q^{n-1} \subset \cdots \subset Q^1 \subset Q^0 = Q$$

with the following properties

(1) *If $P_1 = Q_i$ then $Q_i^k = P_i = Q_i$ for all $k = 1, 2, \ldots n-1$, $i = 1,2$;*
(2) *Q^k is an index pair for all $k = 1, 2, \ldots n-1$;*
(3) *$F(Q_i^k) \cap N \subset Q_i^{k+1}, i = 1,2,\; k = 0, 1, \ldots n-1$.*

Proof. Let Q^k be given by the recurrence formula $Q^0 = Q$, $Q^{k+1} = G(P, Q^k)$, $k = 0, 1, \ldots$ By Lemma 2.13 and induction on k, $\{Q^k\}$ is a decreasing sequence of index pairs containing P and satisfying (1), (2) and (3) for all $k \in \mathbb{Z}^+$. It remains to show that $Q^n = P$ for some n. Indeed, suppose that the inclusion $P \subset Q^k$ is strict for all k, i.e. if $i \in \{1,2\}$ is such $Q_i \backslash P_i \neq \emptyset$ then $Q_i^k \backslash P_i \neq \emptyset$. Let us choose $\sigma(k) \in Q_i^k \backslash P_i$. Then $\sigma(k) \in F(Q_i^{k-1} \cap N)$, so there exists $\sigma(k-1) \in Q_i^{k-1}$ with $\sigma(k) \in F(\sigma(k-1))$. If $\sigma(k-1) \in P_i$ then, by the property (a) of index pair, $\sigma(k) \in P_i$, a contradiction. Therefore $\sigma(k-1) \in Q_i^{k-1} \backslash P_i$. By the reverse reccurrence, one may construct a solution $\sigma_k : [0,k] \to Q_i \backslash P_i \subset Q_i \backslash \operatorname{int} P_i$ such that $\sigma(j) \in Q_i^i \backslash P_i, j = 0, 1, \ldots k, k \in \mathbb{Z}^+$. By Lemma 2.8, $\operatorname{inv} N \subset \operatorname{int}(P_1 \backslash P_2) \subset \operatorname{int} P_1$.

If $i = 2$, we get $\emptyset \neq \operatorname{inv}(Q_2 \setminus \operatorname{int} P_2) \subset \operatorname{inv} Q_2$. On the other hand $\operatorname{inv} Q_2 \subset Q$ and $\operatorname{inv} Q_2 \subset \operatorname{int}(Q_1 \setminus Q_2) \subset Q_1 \setminus Q_2$ implies that $\operatorname{inv} Q_2 = \emptyset$, a contradiction. $\square$

3. Definition of the Conley Index for DMDS

Since now we shall consider dmds $F : X \times \mathbb{Z} \to \mathcal{P}(X)$ such that the generator $F = F^1 : X \to \mathcal{P}(X)$ is *determined by a given morphism.* For the definition of that concept we refer the reader to [7] and [8] (comp. also [13]). Briefly speaking, the class of maps determined by a morphisms is the class of those multivalued maps which admit a representation inducing a homomorphism in cohomology. Any single-valued continous map and any composition of acyclic maps (i.e. usc maps with compact acyclic values) belongs to that class.

If P is an index pair for an isolating neighbourhood $N \subset X$, we let

$$\begin{aligned} S(P) :=\ & (P_1 \cup F(P_2), P_2 \cup F(P_2)) \\ T(P) :=\ & T_N(P) := (P_1 \cup (X \setminus \operatorname{int} N), P_2 \cup (X \setminus \operatorname{int} N)). \end{aligned}$$

Proposition 3.1. *If P is an index pair for N then*

(a) $F(P) \subset S(P) \subset T(P)$;

(b) *The inclusions $i_{P,S(P)}, i_{S(P),T(P)}$ and, consequently, $i_{P,T(P)}$ induce isomorphisms in the Alexander-Spanier cohomology.*

Proof. (a) If $y \in F(P_i)$ then either $y \in P_i$ or $y \in N$. In the second case $y \in F(P_2)$ wich proves the first inclusion. Since $F(P_2) \subset F(P_1)$ and $X \setminus N \subset X \setminus \operatorname{int} N$, the second inclusion follows by the same argument.

(b) By Proposition 2.12 (i), $S_1(P) \setminus S_2(P) = P_1 \setminus P_2$. We also have $T_1(P) \setminus T_2(P) = (P_1 \setminus P_2) \cap \operatorname{int} N = P_1 \setminus P_2$, hence (b) follows from the strong excision property [17]. $\square$

Let now $F_{P,T(P)} : P \to \mathcal{P}(T(P))$ be the restriction of F to domain P and codomain $T(P)$ and let $i_P := i_{P,T(P)}$. The endomorphism $I_P := H^*\left(F_{P,T(P)}\right) \circ H^*(i_P)^{-1}$ of $H^*(P)$ is called the *index map* associated with the index pair P. Let $L(H^*(P), I_P)$ denote the *Leray reduction* of the Alexander-Spanier cohomology for P, as defined in [13]. ($L(H^*(P), I_P)$ is a pair consisting of a graded module over $\mathbb{Z}$ and its endomorphism).

A compact subset K of X is called an *isolated invariant set* if $K = \operatorname{inv} N$ for an isolating neighbourhood N containing K. In such case we say that N is an *isolating neighbourhood of K for F*. The main result of this section is the following.

Theorem 3.2. *Let K be an isolated invariant set. Then $C(K, F) := L(H^*(P), I_P)$ is independent of the choice of an isolating neighbourhood N of K for F and of an index pair P for N.*

The module $C(K,F)$ given by the above theorem (denoted shortly by $C(K)$ if F is clear from the context) is called the *cohomologial Conley index of* K.

Proof of Theorem 3.2 We need to show that if M and N are two isolating neighbourhoods of K, P an index pair for N and Q an index pair for M then $L(H^*(P), I_P) = L(H^*(Q), I_Q)$. The proof will be given in several steps.

Step 1 We consider the following special case

(i) $M = N$,

(ii) $P \subset Q$,

(iii) $F(Q) \subset T_N(P)$,

By (iv), we may consider the map $F_{Q,T(P)} : Q \to \mathcal{P}(T_N(P))$, $F_{Q,T(P)}(x) = F(x)$, and the induced homomorphism $I_{Q,P} := H^*(F_{Q,T(P)}) \circ H^*(i_P)^{-1}$. $I_{Q,P} := H^*(F_{Q,T(P)}) \circ H^*(i_p)^{-1}$. We obtain the commutative diagram

$$\begin{array}{ccc} H^*(P) & \xleftarrow{I_P} & H^*(P) \\ \uparrow{\scriptstyle H*(j)} & \swarrow I_{Q,P} & \uparrow{\scriptstyle H^*(j)} \\ H^*(Q) & \xleftarrow{I_Q} & H^*(Q) \end{array} ,$$

where $j : P \to Q$ is the inclusion which shows that $(H^*(P), I_P)$ and $(H^*(Q), I_Q)$ are linked in the sense of [13]. By [13] $LH^*(j) : L(H^*(Q), I_Q) \to L(H^*(P), I_P)$ is an isomorphism.

Step 2 We lift the assumption (iv). Let $\left\{Q^k\right\}_{k=0,1\ldots n-1}$ be such that the pair of index pairs $Q^{k+1} \subseteq Q^k$ satistfies the assumptions (ii) through (iv), so their corresponding Leray reductions are isomorphic. Since $Q^0 = Q$ and $Q^n = P$, the conclusion follows.

Step 3 We lift the assumption (iii). Put $R_1 := P_1 \cup Q_2, R_2 := P_1 \cap Q_2$. By Proposition 2.12 (iii),(P_1, R_2) and (R_1, Q_2) are index pairs. We have the commutative diagram of inclusions

$$\begin{array}{ccc} (P_1, R_2) & \xrightarrow{j_2} & (R_1, Q_2) \\ \uparrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_3} \\ (P_1, P_2) & \xrightarrow{j} & (Q_1, Q_2) \end{array}$$

It is clear that the pair of index pairs $(P_1, P_2) \subset (P_1, P_2)$ satisfies the assumption (iii) and so does $(R_1, Q_2) \subset (Q_1, Q_2)$, so by Step 2, the inclusions i_1 and i_3 induce isomorphisms of the Leray reductions of the corresponding cohomologies. Since $P_1 \backslash R_2 = P_1 \backslash Q_2 = R_1 \backslash Q_2$, the inclusion j_2 induces an isomorphism in cohomologies by the strong excision property.

Step 4 Now, only (i) is assumed. By Proposition 2.12 (ii), $P \cap Q$ is an index pair, hence the conclusion of Step 3 can be applied to pairs $P \cap Q \subset P$ and $P \cap Q \subset Q$.

Step 5 If $M \neq N$, one may always assume that $M \subset N$ since otherwise $M \cap N$ can be concidered which is also an isolating neighbourhood of K. By Step 4, it is sufficient to show the existence of one index pair P for N and one index pair Q for M such that $L(H^*(P), I_P)$ and $L(H^*(Q), I_Q)$ are isomorphic.

By Theorem 2.6 there exists an index pair P for N such that $P_1 \backslash P_2 \subset \operatorname{int} M \cap \{x \in X : F(x) \subset \operatorname{int} M\}$. It is easily verified that $Q := (M \cap P_1, M \cap P_2)$ is an index pair for M. Since $Q_1 \backslash Q_2 = M \cap (P_1 \backslash P_2) = P_1 \backslash P_2$, the inclusion $Q \subseteq P$ induces an isomorphism in cohomology, by the strong excision property. $\square$

4. Homotopy and additivity properties of the Conley Index.

Let $\Lambda \subset R$ be a compact interval and $F : \Lambda \times X \times \mathbb{Z} \to \mathcal{P}(X)$ an usc mapping with compact values, determined by a given morphism and such that, for each $\lambda \in \Lambda, F_\lambda : X \times \mathbb{Z} \to \mathcal{P}(X)$ given by $F_\lambda(x, n) := F(\lambda, x, n)$ is a dmds. Given a compact subset $N \subset X$ and $\lambda \in \Lambda$, the sets $\operatorname{inv}^{(\pm)} N$ with respect to F_λ are denoted by $\operatorname{inv}^{(\pm)}(N, \lambda)$. We will discuss the following *homotopy property* (called also *continuation property*) of the Conley index:

Theorem 4.1. *Let $\lambda_0 \in \Lambda$ and let N be an isolating neighbourhood for F_{λ_0}. Then*

(a) *N is an isolating neighbourhood for F_λ for all λ sufficiently close to F_{λ_0}.*
(b) *If N is an isolating neighbourhood for F_λ for all λ then $C(\operatorname{inv}(N, \lambda))$ does not depend on λ.*

We prove here the assertion *(a)* only, since the proof of *(b)* is exactly the same as in the single-valued case in [13].

Lemma 4.2. *Let $N \subset X$ be compact. Then the mappings $\lambda \to \operatorname{inv}^+(N, \lambda), \lambda \to \operatorname{inv}^-(N, \lambda)$, and $\lambda \to \operatorname{inv}(N, \lambda), \lambda \in \Lambda$, are usc.*

Proof. We prove the assertion for the first mapping, since the other two proofs are by extending the same argument to negative integers. Suppose that $\lambda \to \operatorname{inv}^+(N, \lambda)$ is not usc at $\lambda_0 \in \Lambda$. Then there exists an open U and a sequence $\lambda_n \to \lambda_0$ such that $\operatorname{inv}^+(N, \lambda_0) \subseteq U$ but $\operatorname{inv}^+(N, \lambda_n) \cap N \backslash U \neq \emptyset$. Let $x_n \in \operatorname{inv}^+(N, \lambda_n) \cap (N \backslash U)$. Since $N \backslash U$ is compact, we may assume that $x_n \to x \in N \backslash U$. In order to achieve a contradiction, we have to show that $x \in \operatorname{inv}^+(N, \lambda_0)$. Indeed, let $\sigma_n : \mathbb{Z}^+ \to N$ be a solution for F_{λ_n} with $\sigma_n(0) = x_n$. Then $\sigma_n(k) \subset \operatorname{inv}^+(N, \lambda_n) \subseteq N \backslash U$ for all $k = 1, 2, \ldots$ We construct a solution $\sigma : \mathbb{Z}^+ \to N \backslash U$ for F_λ by induction on k. Let $\sigma(0) = \lim_n \sigma_n(0) = x$. Let $\sigma(k)$ be constructed for a given k, so that $\sigma(k) = \lim_i \sigma_{n_i}(k)$, where $\{\sigma_{n_i}(k)\}_i$ is a subsequence of $\{\sigma_n(k)\}_n$ convergent in $N \backslash U$. Passing again to a subsequence, we may assume that $\{\sigma_{n_i}(k+1)\}_i$ is convergent. Define $\sigma(k+1)$ to be its limit. Since $\sigma_n(k+1) \in F(\lambda_n(k))$ for all n, the closed graph property of F implies that $\sigma(k+1) \in F(\lambda, \sigma(k))$. $\square$

Proof of Theorem 4.1 By the compactness of N, the condition (2.1) implies that

$$B_{\operatorname{diam}_N F_{\lambda_0}+3\varepsilon}(\operatorname{inv}(N,\lambda_0)) \subset \operatorname{int} N$$

for some $\varepsilon > 0$. Since F is usc., $F_\lambda(x) \subset B_\varepsilon(F_{\lambda_0}(X))$ for all λ close to λ_0 and all $x \in N$. Again by compactness of N,

$$\operatorname{diam}_N F_\lambda < \operatorname{diam}_N F_{\lambda_0} + 2\varepsilon$$

for all λ close to λ_0. By Lemma 4.2, $\operatorname{inv}(N,\lambda) \subset B_\varepsilon(\operatorname{inv}(N,\lambda_0))$ for all λ close to λ_0 and we get

$$\begin{aligned} B_{\operatorname{diam}_N F_\lambda}(\operatorname{inv}(N,\lambda)) &\subset B_{\operatorname{diam}_N F_{\lambda_0}+2\varepsilon}(B_\varepsilon(\operatorname{inv}(N,\lambda_0))) \\ &= B_{\operatorname{diam}_N F_{\lambda_0}+3\varepsilon}(\operatorname{inv}(N,\lambda_0)) \\ &\subset \operatorname{int} N. \quad \square \end{aligned}$$

By the same arguments as in [13] one proves the folowing *additivity property* of the Conley index:

Theorem 4.3. *Let $K \subset X$ be an isolated invariant set for a dmds $F : X \times \mathbb{Z} \to \mathcal{P}(X)$, wich is a disjoint sum of two other isolated invariant sets K_1, K_2. Then $C(K) = C(K_1) \times C(K_2)$.* $\square$

5. Examples of applications.

The first two examples are supposed only to illustrate the main concepts introduced in this paper.

Example 5.1. Assume $\epsilon \geq 0$. Define $F : \mathbb{R}^2 \to \mathcal{P}(\mathbb{R}^2)$ by

$$F_\epsilon(x,y) := \{(u,v) : |\frac{x}{2} - u| \leq \epsilon, |2y - v| \leq \epsilon\}.$$

and consider the set $N := \{(u,v) : |u| \leq 1, |v| \leq 1\}$. One can easily compute that

$$F_\epsilon^{-1}(N) \cap N \cap F_\epsilon(N) = \{(u,v) : |u| \leq \frac{1}{2} + \epsilon, |v| \leq \frac{1+\epsilon}{2}\},$$

hence

$$\operatorname{dist}(F_\epsilon^{-1}(N) \cap N \cap F_\epsilon(N), \operatorname{bd} N) > \operatorname{diam}_N F_\epsilon$$

for $\epsilon < \frac{1}{6}$. Thus for such an ϵ the set N is an isolating block with respect to F_ϵ. Taking $P_1 := N$, $P_2 := \{(x,y) : |y| \geq \frac{1-\epsilon}{2}\} \cap N$ one can show that (P_1, P_2) is an index pair for F_ϵ in N and compute that the Conley index is

$$\operatorname{Con}(F_\epsilon, N) = 0 \oplus (\mathbb{Z}, \operatorname{id}) \oplus 0 \oplus \cdots .$$

Example 5.2. Define $G_\epsilon : \mathbb{C} \to \mathbb{C}$ by

$$G_\epsilon(z) := B_{\epsilon|z|}(z^2)$$

and consider the set $M := \{z : \frac{1}{2} \le |z| \le 2\}$. One can show that

$$G_\epsilon^{-1}(M) \cap M \cap G_\epsilon(M) = \{z : \frac{\sqrt{\epsilon^2+2}-\epsilon}{2} \le |z| \le \frac{\sqrt{\epsilon^2+8}+\epsilon}{2}\}$$

Thus M is an isolating block with respect to G_ϵ for $\epsilon < \frac{\sqrt{161}-9}{6}$. One can check that $(M, \emptyset)$ is an index pair for G_ϵ in M, thus concluding that

$$\mathrm{Con}(G_\epsilon, M) = (\mathbb{Z}, \mathrm{id}) \oplus (\mathbb{Z}, \mathrm{id}) \oplus 0 \oplus 0 \cdots .$$

As we mentioned in the introduction, the main application of the theory is the rigorous computation of the Conley index of a map on the basis of numerical computations approximating the map. To be more precise, assume that we have an L-Lipschitz continuous function $f : \mathbb{R}^m \to \mathbb{R}^m$ and $N \subset \mathbb{R}^m$ is a compact subset. Furthermore, assume that there is a family of finite subsets $\bar{N}_n \subset N$ such that for any $x \in N$ $\mathrm{dist}(x, \bar{N}_n) \le 1/n$ and for any $n \in N$ and $\bar{x} \in \bar{N}_n$ we can find numerically an approximate value $\bar{f}_n(\bar{x})$ such that $\mathrm{dist}(f(\bar{x}), \bar{f}_n(\bar{x})) \le 1/n$. We define the n-th multivalued approximation of f by

$$F_n : N \ni x \to \mathrm{conv} \bigcup \{B_r(\bar{f}(\bar{x})) : \mathrm{dist}(x, \bar{x}) = \mathrm{dist}(x, \bar{N}_n)\},$$

where $r = \frac{1+L}{n}$. It is an easy excercise to prove the following theorem.

Theorem 5.3. *(comp.* [14]*)*

(a) *For any $n \in \mathbb{N}$ the map f is a selector of F_n.*
(b) *F_n is an upper semicontinuous convex-valued (in particular admissible) map.*
(c) *If N is an isolating neighbourhood for f then N is an isolating neighbourhood for F_n with n sufficiently large.*

On the other hand, if F is a dmds with convex values and f is its continuous selector, it is easy to verify that

$$F_\lambda(x) := \lambda f(x) + (1-\lambda)F(x)$$

satisfies the assumption (b) of Theorem 4.1. Therefore we obtain the following corollary of that theorem

Theorem 5.4. *(comp.* [14]*) Assume N is an isolating neighbourhood for a convex-valued dmds F. Then N is an isolating neighbourhood for any its continuous selector f, and the Conley indexes of f and F in N coincide.*

For simplicity of the presentation we assumed that F is convex-valued but it is sufficient here to assume that F is acyclic (c.f. [8], Corollary 3.2).

Thus the above theorem enables carrying over the information about the Conley index from a numerically computed multivalued approximation of f to the map f itself. Examples of applications in this direction are presented in [11, 14].

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