Shape Index and Other Indices of Conley Type
for Local Maps on Locally Compact Hausdorff Spaces

by

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Abstract. We present a scheme for constructing various Conley indices for locally defined maps. In particular we extend the shape index of Robbin and Salamon to the case of a locally defined map in a locally compact Hausdorff space. We compare the shape index with the cohomological Conley index for maps. We also prove the commutativity property of the Conley index, which is analogous to the commutativity property of the fixed point index.

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Introduction. The shape index is a time-discrete analog of Conley's homotopy index for flows. It was constructed by Robbin and Salamon [RS] for isolated invariant sets of a diffeomorphism on a smooth manifold. In [Mr1] the author presented a cohomological Conley index for isolated invariant sets of homeomorphisms (see also [MR]). Its construction was based on a functor, called the Leray functor and introduced in [Mr1, Sect. 4]. It turns out that there are at least three other functors which can be used in the construction instead of the Leray functor. They provide various Conley indices but with the same basic properties. One of such functors is the inverse limit functor, which can be used to construct the shape index.

In the present paper, we propose a general scheme for constructing Conley indices. This enables us to unify the results in [RS] and [Mr1], to get rid of smoothness and injectivness in the construction of the shape index and to compare the shape index with the cohomological Conley index. The scheme provides also other Conley indices: the homology index, the homotopy group index, the shape group index.

Contrarily to [RS] and [Mr1], in this paper we work with locally defined maps. This is important beacuse of forthcoming applications to differntial equations, where one has to deal with t-translations or Poincare maps which often are defined only in some open subset of the given space.

We prove basic properties of the Conley index: the Ważewski property, the homotopy property, the additivity property and the commutativity property. The commutativity property for the Conley index, which is an analog of the commutativity property of the fixed point index (see [Do, Sect. VI.5.9]), seems to appear here for the first time.

The contents of the paper are as follows. Main results are summarized in the first section. The shape index is introduced in the second section and various algebraic Conley indices in the third section. Section 4 presents the adaptation of the construction of index pairs from [Mr1] to the case of a map. In Section 5 we make the division of compact spaces functorial by introducing the category of pairs and the quotient functor. This is used in the following section to construct the Conley index. The remaining two sections contain proofs of the properties of the Conley index.

In the paper we follow the notations and conventions introduced in [Mr1].

1. Main results. In this section, we give a brief overlook of the results of the paper. Details are postponed to the following sections.

Let X denote a fixed, locally compact Hausdorff space. By a local discrete semidynamical system on X we mean a continuous map $f : X \multimap X$ defined on some open subset of X. We say that the function $\sigma : \mathbb{Z} \to X$ is a (full) solution to f through x in $N \subset X$ if $f(\sigma(i)) = \sigma(i+1)$ for all $i \in \mathbb{Z}$, $\sigma(0) = x$ and $\sigma(i) \in N$ for all $i \in \mathbb{Z}$. The invariant part of $N \subset X$ (with respect to f) is defined as the set of all $x \in N$ which admit a solution to f through x in N. It will be denoted by $\operatorname{Inv}(N, f)$ or $\operatorname{Inv} N$, if f is clear from context.

The set K is said to be invariant if $f(K) = K$. This is easily seen to be equivalent to $K = \operatorname{Inv}(K, f)$. K is called isolated invariant, if it admits a compact neighborhood N such that $K = \operatorname{Inv} N$. The neighborhood N is then called an isolating neighborhood of K.

The set $A \subset N$ is called positively invariant with respect to N and f if $A \cap f^{-1}(N) \subset f^{-1}(A)$.

In order to assign to each isolated invariant set an index, we need the notion of the index pair. There are various concepts of index pairs. A very weak notion of the index pair was proposed by Robbin and Salamon [RS, Def.4.1]. The following definition is a straightforward adaptation of the original definition by Conley [Co, Def.III.4.1].

Definition 1.1. The pair $P = (P_1, P_2)$ of compact subsets of N will be called an index pair of K in N (with respect to f) iff the following three conditions are satisfied.

(1.1) P_1, P_2 are positively invariant with respect to N,

(1.2) $P_1 \setminus P_2 \subset f^{-1}(N)$,

(1.3) $K \subset \operatorname{int}(P_1 \setminus P_2)$.

The index pair introduced in [Mr1, Def.2.1] is stronger. Index pairs of this kind are called in this paper *strong index pairs* (see Def. 4.1).

As in [Mr1] we do not assume that $P_2 \subset P_1$, i.e., that P is a topological pair. However, it is straightforward to verify that if (P_1, P_2) is an index pair, so is $(P_1, P_1 \cap P_2)$.

The family of all index pairs in N will be denoted by $\mathrm{IP}(N, f)$ or simply by $\mathrm{IP}(N)$.

Theorem 1.2. *Every isolating neighborhood admits an index pair.*

(The proof will be given in Sect. 4. For similar theorems in different settings see [RS, Mr1, MR]).

Let P_1/P_2 denote the quotient space. We will consider it as an object in Comp_*, the category of pointed Hausdorff compact spaces, by assuming that the point distinguished in P_1/P_2 is P_2 collapsed to a point.

The map f induces a continuous map $f_P : P_1/P_2 \to P_1/P_2$ called the index map (see Def. 6.1).

It turns out to be natural to consider each index pair with respect to f together with its index map f_P. For this reason, together with each category $\mathcal{E}$ we consider the category of endomorphisms of $\mathcal{E}$ denoted by $\mathrm{Endo}(\mathcal{E})$. This concept was introduced in [Mr1, Sect. 1] in the special case of the category of graded moduli. The objects of $\mathrm{Endo}(\mathcal{E})$ are pairs (A, a), where $A \in \mathcal{E}$ and $a \in \mathcal{E}(A, A)$ is a distinguished endomorphism of A. The set of morphisms from $(A, a) \in \mathcal{E}$ to $(B, b) \in \mathcal{E}$ is the subset of $\mathcal{E}(A, B)$ consisting of exactly those morphisms $\varphi \in \mathcal{E}(A, B)$ for which $b\varphi = \varphi a$. We write $\varphi : (A, a) \to (B, b)$ to denote that φ is a morphism from (A, a) to (B, b) in $\mathrm{Endo}(\mathcal{E})$. We define the category of automorphisms of $\mathcal{E}$ as the full subcategory of $\mathrm{Endo}(\mathcal{E})$ consisting of pairs $(A, a) \in \mathrm{Endo}(\mathcal{E})$ such that $a \in \mathcal{E}(A, A)$ is an automorphism, i.e., both an endomorphism and an isomorphism in $\mathcal{E}$. The category of automorphisms of $\mathcal{E}$ will be denoted by $\mathrm{Auto}(\mathcal{E})$. There is a functorial embedding

$$\mathcal{E} \ni A \to (A, \mathrm{id}_A \in \mathrm{Auto}(\mathcal{E})),$$

$$\mathcal{E}(A,B) \ni \varphi \to \varphi \in \mathrm{Auto}(\mathcal{E})(A,B)\,,$$

hence we can consider the category $\mathcal{E}$ as a subcategory of $\mathrm{Auto}(\mathcal{E})$.

Thus, we can write $(P_1/P_2, f_P) \in \mathrm{Endo}(\mathrm{Comp}_*)$.

Assume $\mathcal{C}$ is a full subcategory of $\mathrm{Endo}(\mathcal{E})$ and $F : \mathcal{C} \to \mathrm{Auto}(\mathcal{E})$ is a functor. Let $(A,a) \in \mathcal{C}$. Then $F(A,a)$ is an object of $\mathrm{Auto}(\mathcal{E})$. Let a' denote the automorphism distinguished in $F(A,a)$. Obviously $a : (A,a) \to (A,a)$ is a morphism in $\mathrm{Endo}(\mathcal{E})$ and since $\mathcal{C}$ is a full subcategory of $\mathrm{Endo}(\mathcal{E})$ it is also a morphism in $\mathcal{C}$. Hence $F(a)$ is defined and it is a morphism from $F(A,a)$ to $F(A,a)$ in $\mathrm{Auto}(\mathcal{E})$. However, it need not be $F(a) = a'$ in general. $F(a)$ need not be even an isomorphism in $\mathrm{Auto}(\mathcal{E})$.

Definition 1.3. We will say that $F : \mathcal{C} \to \mathrm{Auto}(\mathcal{E})$ is normal, if for each $(A,a) \in \mathcal{C}$ the morphism $F(a)$ is equal to the automorphism distinguished in $F(A,a)$.

The main reason for introducing normal functors is the following theorem, which is essential in the construction of the Conley index. It can be proved as an easy exercise (comp. [Mr2, Th.3.1]).

Theorem 1.4. *Assume $\mathcal{C}$ is a full subcategory of $\mathrm{Endo}(\mathcal{E})$, $F : \mathcal{C} \to \mathrm{Auto}(\mathcal{E})$ is a normal functor, $(A,a), (B,b) \in \mathcal{C}$ and $\varphi \in \mathcal{E}(A,B)$, $\psi \in \mathcal{E}(B,A)$ are such that $a = \varphi\psi$, $b = \psi\varphi$. Then φ, ψ are also morphisms in $\mathrm{Endo}(\mathcal{E})$ and we have the commutative diagram*

$$(1.4)\qquad \begin{tikzcd} F(A,a) \arrow[r, "F(a)"] \arrow[d, "F(\varphi)"'] & F(A,a) \arrow[d, "F(\varphi)"] \\ F(B,b) \arrow[r, "F(b)"'] \arrow[ur, "F(\psi)"] & F(B,b) \end{tikzcd}$$

in $\mathrm{Auto}(\mathcal{E})$, in which all morphisms are isomorphisms.

There is a scheme for producing normal functors. To explain it, assume $\mathcal{C} \subset \mathrm{Endo}(\mathcal{E})$ is a full subcategory and $L : \mathcal{C} \to \mathcal{E}$ is a functor. Assume (A,a), $(B,b) \in \mathrm{Endo}(\mathcal{E})$ and

$\varphi : (A, a) \to (B, b)$ is a morphism in $\mathrm{Endo}(\mathcal{E})$. Put

$$L'(A, a) := (L(A, a), L(a)), \tag{1.5}$$

$$L'(\varphi) := L(\varphi). \tag{1.6}$$

Theorem 1.5. *(comp. [Mr2]). If for every $(A, a) \in \mathcal{C}$ $L(a)$ is an isomorphism in $\mathcal{E}$ then formulae (1.5-1.6) define a normal functor $L' : \mathcal{C} \to \mathrm{Auto}(\mathcal{E})$.*

There are trivial examples of normal functors $F : \mathrm{Endo}(\mathcal{E}) \to \mathrm{Auto}(\mathcal{E})$: If $O \in \mathcal{E}$ is the zero object in $\mathcal{E}$, then one can define the zero functor from $\mathrm{Endo}(\mathcal{E})$ to $\mathrm{Auto}(\mathcal{E})$ by assigning (O, id) to each object $(A, a) \in \mathrm{Endo}(\mathcal{E})$ and the identity map in O to each morphism $\varphi : A \to B$ in $\mathcal{E}$. It is straightforward to verify that the zero functor is a normal functor. We are interested, however, in less trivial examples.

A good condition for nontriviality, which is also useful in applications, is to assume that F restricted to the subcategory $\mathrm{Auto}(\mathcal{E})$ is naturally equivalent to the identity functor. Hence we introduce the following.

Definition 1.6. Assume $\mathcal{B}$ is a category and $\mathcal{B}_0$ is its subcategory. The functor $F : \mathcal{B} \to \mathcal{B}_0$ will be called a retractor iff F restricted to $\mathcal{B}_0$ is naturally equivalent to the identity functor on $\mathcal{B}_0$.

An example of a normal retractor is the Leray functor introduced in [Mr1, Sect. 4].

Assume now that $T : \mathrm{Comp}_* \to \mathcal{E}$ is a co- or contra-variant functor, which is homotopy invariant. It extends in a natural way to a functor $T : \mathrm{Endo}(\mathrm{Comp}_*) \to \mathrm{Endo}(\mathcal{E})$ denoted with the same letter. Assume also that $\mathcal{C} \subset \mathrm{Endo}(\mathrm{Comp}_*)$ is a subcategory such that $T(\mathrm{Endo}(\mathrm{Comp}_*)) \subset \mathcal{C}$. Let $L : \mathcal{C} \to \mathrm{Auto}(\mathcal{E})$ be a normal functor. Then the composite functor $LT := L \circ T : \mathrm{Endo}(\mathrm{Comp}_*) \to \mathrm{Auto}(\mathcal{E})$ is defined.

Theorem 1.7. *Assume K is an isolated invariant set with respect to f. Then*

$$LT(P_1/P_2, f_P) \text{ and } LT(Q_1/Q_2, f_Q)$$

are isomorphic objects in $\mathrm{Auto}(\mathcal{E})$ *for any isolating neighborhoods N, M of K and* $P \in \mathrm{IP}(N)$, $Q \in \mathrm{IP}(M)$.*(for proof see Sect. 6)*

Theorems 1.2 and 1.7 allow us to put the following

Definition 1.8. The common value $LT(P_1/P_2, f_P)$ for all index pairs P of K will be called the (L,T)-Conley index of K and denoted by $C_{L,T}(K,f)$. If this causes no misunderstanding, we will write $C(K,f)$ or simply $C(K)$.

The empty pair $(\emptyset, \emptyset)$ is obviously an index pair of the empty set. Hence we get the following important property of the Conley index.

Proposition 1.9 (Ważewski Property). If T preserves zero objects and L is a retractor, then $C(\emptyset) = 0$, i.e. $C(K) \neq 0$ implies $K \neq \emptyset$. QED

We call it the Ważewski property because in the case of a flow it can be considered as a reformulation of the Ważewski Retract Theorem (see [Co, II.2.3] or [Wa]).

Theorem 1.10 (Homotopy Property). *Assume $f : \Lambda \times X \multimap X$ is a continuous map, $\Lambda \subset \mathbb{R}$ is a compact interval and N is an isolating neighborhood with respect to each partial map*

$$f_\lambda : X \ni x \multimap f(\lambda, x) \in X\,.$$

Then $C(\mathrm{Inv}(N, f_\lambda), f_\lambda)$ does not depend on $\lambda \in \Lambda$ (for proof see Sect. 7).

Theorem 1.11 (Additivity Property). *Assume an isolated invariant set K is a disjoint sum of two other isolated invariant sets K_1, K_2. If T and L preserve coproducts then $C(K)$*

is the coproduct of $C(K_1)$ and $C(K_2)$. If T sends coproducts into products and L preserves products then $C(K)$ is the product of $C(K_1)$ and $C(K_2)$ (for proof see Sect. 8).

Theorem 1.12 (Commutativity Property). *Assume Y is another locally compact Hausdorff space and $\varphi : X \multimap Y$, $\psi : Y \multimap X$ are maps with open domains, $f = \psi\varphi$, $g := \varphi\psi$. If $K \subset X$ is an isolated invariant set with respect to f then $\varphi(K)$ is an isolated invariant set with respect to g and $C(K, f) = C(\varphi(K), g)$ (for proof see Sect. 8).*

As an example of a straightforward application of the commutativity property, we have the following theorem.

Theorem 1.13. *Assume $f(X) \subset A$, where A is a locally compact subset of X. If K is an isolated invariant set with respect to f then K is an isolated invariant set with respect to $f\,|_A$ and $C(K, f) = C(K, f\,|_A)$.*

Proof: Let $i : A \to X$ denote the inclusion. Take $\varphi := i$, $\psi := f$ and apply the commutativity property. QED

2. The Shape Index. The notion of shape was invented by K. Borsuk [Bo1, Bo2] as a modification of the homotopy type (see also Borsuk's book [Bo3]). Homotopy theory behaves well if the spaces are locally nice like ANR's or polyhedra. Shape theory takes into account only the global properties of spaces in such a way that it is equivalent to homotopy theory when restricted to ANR's and has some nice properties, which fail in case of homotopy theory.

The precise definition of shape, regardless of approach, is rather complicated. From our point of view it is essential to understand only the main similarities and differences between shape and homotopy. A very concise survey of shape theory is presented in [Ma] and we follow here that exposition. We recommend the book of Mardesic and Segal [MS] to readers interested in details.

Roughly speaking, shape theory for compact Hausdorff spaces consists of a certain new category, called the shape category and denoted by Sh and a covariant functor S : Comp $\to$ Sh, called the shape functor. The shape category has compact Hausdorff spaces as objects and some modification of homotopy classes of continuous maps, called shape maps, as morphisms. The shape functor keeps objects fixed and sends each continuous map $f : X \to Y$ to some shape map $Sf : X \to Y$, called the shape map generated by f (though, contrary to the homotopy theory, not every shape map has to be generated by some f). Two spaces are said to have the same shape if they are isomorphic in the shape category. It is obviously an equivalence relation. The equivalence class of a given space X in this relation is called its shape.

The definition of pointed shape is based on a similar scheme (see [MS,I.4.3]). Hence, one has a pointed shape category Sh_* with pointed compact Hausdorff spaces as objects and a covariant functor $S : \mathrm{Comp}_* \to \mathrm{Sh}_*$.

The shape (pointed shape) functor has the following essential properties (see [MS, Chpt. I,Sec .2,Th. 4 and Sec. 5, Cor. 5] or [Bo3, Chpt. XII, Sec. 1]).

(2.1) S is homotopy invariant, i.e. shape maps generated by (point-preserving) homotopic maps are equal.

(2.2) Every shape map into an ANR is generated by a (point-preserving) continuous map unique up to a (point-preserving) homotopy. This means that shape is equivalent to the homotopy type on ANR's.

(2.3) S is continuous with respect to inverse limits, i.e. it sends inverse limits in Comp (Comp_*) into inverse limits in Sh (Sh_*).

It is the third property, which makes shape essential in the construction of an analog for maps of Conley's homotopy index for flows. Note that the homotopy functor lacks this property (see [MS, Chpt. I.5, Ex. 2]), hence, by (2.2), the shape functor can be viewed as a moderate modification of the homotopy functor which ensures the continuity property.

The continuity property allows us to construct the necessary normal functor. The construction is quite general, so we take an arbitrary category $\mathcal{E}$. To each object $(A,a) \in \mathrm{Endo}(\mathcal{E})$ we assign an inverse system of morphisms

$$\ldots \xrightarrow{a} A \xrightarrow{a} A \xrightarrow{a} A\,. \tag{2.4}$$

If (2.4) admits the inverse limit, it will be called the inverse limit of (A,a).

Denote by $\mathrm{Endo}_i(\mathcal{E})$ the full subcategory of $\mathrm{Endo}(\mathcal{E})$ consisting of those objects $(A,a) \in \mathrm{Endo}(\mathcal{E})$ for which the sequence (2.4) admits the inverse limit.

It is straightforward to verify that if $(A,a) \in \mathrm{Auto}(\mathcal{E})$ then A is the inverse limit of (A,a). Hence, we have the following.

Proposition 2.1. $\mathrm{Auto}(\mathcal{E})$ is a full subcategory of $\mathrm{Endo}_i(\mathcal{E})$. QED

Consider (A,a), $(B,b) \in \mathrm{Endo}_i(\mathcal{E})$ and denote the corresponding inverse limits by A', B' respectively. Assume $\varphi : (A,a) \to (B,b)$ is a morphism in $\mathrm{Endo}_i(\mathcal{E})$. Let $\varphi' : A' \to B'$ denote the map induced by φ (the inverse limit of $\{\varphi\}$).

It is straightforward to verify that

$$(A,a) \to A', \quad \varphi \to \varphi'$$

define a functor $Li : \mathrm{Endo}_i(\mathcal{E}) \to \mathcal{E}$, which will be called the inverse limit functor. In the dual way one defines the direct limit functor $Ld : \mathrm{Endo}_d(\mathcal{E}) \to \mathcal{E}$. By Theorem 1.5 there are also associated normal functors $LI : \mathrm{Endo}_i(\mathcal{E}) \to \mathrm{Endo}(\mathcal{E})$, $LD : \mathrm{Endo}_d(\mathcal{E}) \to \mathrm{Endo}(\mathcal{E})$.

Theorem 2.2. $LI : \mathrm{Endo}_i(\mathcal{E}) \to \mathrm{Auto}(\mathcal{E})$, $LD : \mathrm{Endo}_d(\mathcal{E}) \to \mathrm{Auto}(\mathcal{E})$ *are normal retractors.*

Proof: Fix $(A,a) \in \mathrm{Endo}(\mathcal{E})$, put $A' := Li(A,a)$ and $a' := Li(a)$. We will first show that $LI(A,a) \in \mathrm{Auto}(\mathcal{E})$, i.e. that a' is an isomorphism in $\mathcal{E}$.

A', as the inverse limit of (A, a), admits a cone $\{a_i : A' \to A\}$ with the universal factorization property. The family of morphisms $\{s_i := a_{i+1} \mid i \in \mathbb{N}\}$ is also a cone, hence the universal factorization property implies that there exists a morphism $u : A' \to A'$ such that $s_i = a_i u$ for $i \in \mathbb{N}$. Observe that

$$a_i u a' = s_i a' = a_{i+1} a' = a a_{i+1} = a_i \,.$$

Hence the family $\{a_i\}$ factorizes both through ua' and identity. The uniqueness of factorization implies $ua' = \mathrm{id}$. Similarly,

$$a_i a' u = a a_i u = a s_i = a a_{i+1} = a_i$$

and also $a'u = \mathrm{id}$. This shows that a' is an isomorphism, i.e. LI is indeed a functor from $\mathrm{Endo}_i(\mathcal{E})$ into $\mathrm{Auto}(\mathcal{E})$. It is now straightforward to verify that LI is a retractor. Similarly one proves that LD is a retractor. QED

By (2.1), S is homotopy invariant. Moreover, (2.3) implies that

$$S(\mathrm{Endo}(\mathrm{Comp}_*)) \subset \mathrm{Endo}_i(\mathrm{Sh}_*)\,.$$

Hence (LI, S)-Conley index $C_{LI,S}(K, f)$ makes sense. It will be called the shape index. By its very definition, the shape index consists of some shape and a distinguished shape isomorphism class. The two parts of the index will be denoted by $s(K)$ and $si(K)$ respectively. Hence

$$C_{LI,S}(K) = (s(K), si(K))\,.$$

Our shape index generalizes the shape index of Robbin and Salamon in two directions: 1. because of the distinguished shape isomorphism class it carries more information; 2. it is defined for maps which are neither smooth nor injective nor globally defined. More precisely, we have the following:

Proposition 2.3. Assume X is a smooth manifold, $f : X \to X$ is a diffeomorphism and K is an isolated invariant set with respect to f. Then $s(K, f)$ is just the shape index of K in the sense of Robbin and Salamon.

Proof: The shape index of Robbin and Salamon is, by definition, the shape of the one-point compactification of the unstable manifold of K with some special topology (see [RS, Def. 7.4, Def. 9.1]). This compactification is homeomorphic, by [RS, Th. 3.7] to the inverse limit of the sequence

$$\dots \xrightarrow{f_P} P_1/P_2 \xrightarrow{f_P} P_1/P_2 \xrightarrow{f_P} P_1/P_2\,,$$

where (P_1, P_2) is any index pair in the sense of Robbin and Salamon (see [RS, Def..1]) and f_P is the map induced by f. However, by [RS, Cor. 4.4], index pair in our sense is a special case of index pair in the sense of [RS, Def. 5.1]. This completes the proof. QED

Finally, let us mention the following property of shape, which is an elementary exercise in shape theory.

(2.5) The pointed shape functor sends zero objects in Top_* into zero objects in Sh_*.

Hence we have the following theorem.

Theorem 2.4. *The shape index satisfies the Ważewski, homotopy and commutativity properties.*

The shape functor preserves coproducts, however it is not clear whether the inverse limit functor does the same (certainly it preserves products). Hence we pose the following problem.

Problem. Does the shape index have the additivity property?

Another problem is whether it is possible to construct the shape index in non-locally compact spaces like the cohomological Conley index in [MR]. The main obstacle is that the shape theory for non-compact spaces is less satisfactory. In particular, the continuity property seems to fail (comp. [MS, Chpt. I, Sec. 6, Ex. 1]).

3. Algebraic Conley indices. In this paragraph, we assume that $\mathcal{E}$ denotes either the category of groups or the category of (graded) moduli over a fixed ring Ξ. We also assume that $T : \mathrm{Comp}_* \to \mathcal{E}$ is a homotopy invariant, covariant or contravariant functor. Basic examples, which we have in mind, are: various homology and cohomology functors, n-th homotopy group functor, n-th shape group functor (see [MS, Chpt. II, Sect. 3.3]).

In order to define the Conley index by means of T we need normal functors L : $\mathrm{Endo}(\mathcal{E}) \to \mathrm{Auto}(\mathcal{E})$. The inverse limit functor LI defined in the previous section is an example, because, with our choice of $\mathcal{E}$, $\mathrm{Endo}_i(\mathcal{E}) = \mathrm{Endo}(\mathcal{E})$. Another example is LD. We will present two more examples.

First we recall the concept of the generalized kernel (see [Le]). Assume an endomorphism $a : A \to A$ of $A \in \mathcal{E}$ is given. Put

$$\mathrm{gker}(a) := \bigcup \{\ker(a^n) \mid n \in \mathbb{N}\}\,.$$

The dual notion is the generalized image of a:

$$\mathrm{gim}(a) := \bigcap \{\mathrm{im}(a^n) \mid n \in \mathbb{N}\}\,.$$

In some sense (see [Mr2, Th.5.11]), the role dual to $\mathrm{gker}(a)$ plays the set

$$\mathrm{sim}(a) := \{x \in A \mid \exists \{x_n\}_{n=0,\infty} \subset A \ s.t.\ a(x_{n+1}) = x_n \text{ for } n \in \mathbb{N}, \quad x_0 = x\}\,,$$

which will be called the sequential image of a.

We define the category $\mathrm{Mono}(\mathcal{E})$ as the full subcategory of $\mathrm{Endo}(\mathcal{E})$ which objects have a monomorphism as the distinguished endomorphism. Similarly, by distinguishing epimorphisms, we define the category $\mathrm{Epi}(\mathcal{E})$.

Assume (A, a), $(B, b) \in \mathrm{Endo}(\mathcal{E})$ and $\varphi : (A, a) \to (B, b)$ is a morphism in $\mathrm{Endo}(\mathcal{E})$. Put

$$Lm(A, a) := A/\,\mathrm{gker}(a)\,, \tag{3.1}$$

$$Lm(\varphi) := (A/\operatorname{gker}(a) \ni [x] \to [\varphi(x)] \in B/\operatorname{gker}(b)), \tag{3.2}$$

$$Le(A,a) := \operatorname{gim}(a), \tag{3.3}$$

$$Le(\varphi) := (\operatorname{gim}(a) \ni x \to \varphi(x) \in \operatorname{gim}(a)), \tag{3.4}$$

$$Ls(A,a) := \operatorname{sim}(a), \tag{3.5}$$

$$Ls(\varphi) := (\operatorname{sim}(a) \ni x \to \varphi(x) \in \operatorname{sim}(b)). \tag{3.6}$$

One can easily verify that formulae (3.1-3.6) define three functors Lm, Le, $Ls : \operatorname{Endo}(\mathcal{E}) \to \mathcal{E}$. By Theorem 1.5 we have also functors

$$LM := (Lm)', LE := (Le)', LS := (Ls)' : \operatorname{Endo}(\mathcal{E}) \to \operatorname{Endo}(\mathcal{E}).$$

It is an easy exercise to prove the following proposition.

Proposition 3.1. The composite functors $LMS := LM \circ LS$ and $LSM := LS \circ LM$ are normal retractors from $\operatorname{Endo}(\mathcal{E})$ to $\operatorname{Auto}(\mathcal{E})$. They preserve finite products and coproducts.

Examples in [Mr2, Sect. 6] show that the four normal retractors LI, LD, LMS, LSM are all different. Nevertheless, under some restrictions, they coincide, as shows the following.

Theorem 3.2 (see [Mr2, Th. 6.1]). *Assume $\mathcal{E}$ is the category of vector spaces over a fixed field and $(A,a) \in \operatorname{Endo}(\mathcal{E})$ is a finite dimensional vector space with a distinguished endomorphism. Then $LI(A,a)$, $LD(A,a)$, $LMS(A,a)$, $LSM(A,a)$ are all isomorphic.*

Remark 3.3. It should be noted that $Le(a)$ for an endomorphism $a : A \to A$ need not be an epimorphism. This causes the functor LE to be considered only as a functor from $\mathrm{Endo}(\mathcal{E})$ into $\mathrm{Endo}(\mathcal{E})$. However, it is obvious that LE and LS are equal on the subcategory $\mathrm{Mono}(\mathcal{E})$. In particular $LE \circ LM = LS \circ LM$. The composite functor $LE \circ LM$ was introduced in [Mr1] and called the Leray functor, since the idea of a generalized kernel comes from the paper [Le] of Leray.

Combining various homotopy invariant functors with the above retractors, we obtain various Conley indices. As an example consider the n-th homotopy group functor π_n.

Proposition 3.4. The (LSM, π_n)-Conley index is well defined. It has all four properties considered in Sect. 4.

We finish this section with the following.

Proposition 3.5. The cohomological Conley index introduced in [Mr1] coincides with the (LSM, H^*)-Conley index, where H^* stands for the Alexander-Spanier cohomology.

Proof: Assume K is an isolated invariant set and P is an index pair with respect to K. Put $Q_1 := P_1 \cup f(P_2)$, $Q_2 := P_2 \cup f(P_2)$. We have the following commutative diagram.

$$\begin{tikzcd}
H^*(P_1, P_2) & H^*(Q_1, Q_2) \arrow[l, "f_0"'] \arrow[r, "i"] & H^*(P_1, P_2) \\
H^*(P_1/P_2, [P_2]) \arrow[u] \arrow[d] & H^*(Q_1/Q_2, [Q_2]) \arrow[l, "f_1"] \arrow[r] \arrow[u] & H^*(P_1/P_2, [P_2]) \arrow[u] \arrow[d] \\
H^*(P_1/P_2, [P_2]) & & H^*(P_1/P_2, [P_2]) \arrow[ll, "f_2"]
\end{tikzcd}$$

in which f_0, f_1, f_2 are induced by f and others are induced either by inclusions or projections. All vertical arrows and the arrow marked with i are isomorphisms. The map $f_0 \circ i^{-1}$ is the index map used in [Mr1] to construct the Conley index and denoted there by I_P, whereas $f_2 = H^*(f_P)$. The above diagram shows that $(H^*(P), I_P)$ and $H^*(P_1/P_2, [P_2], f_P)$ are isomorphic objects in $\mathrm{Endo}(\mathcal{E})$. We have also

$$LSM(H^*(P_1/P_2, [P_2], f_P) \sim LSM(H^*(P), I_P) = (LE \circ LM)(H^*(P), I_P).$$

The last equality follows from Remark 3.3. Since $LE \circ LM$ is the Leray functor used in [Mr1], our assertion is proved. QED

4. Construction of index pairs for the discrete case. The construction of index pairs in [Mr1] can be easily adapted to maps in locally compact Hausdorff spaces. Here are the main differences.

For $n \in \mathbb{Z}^+ \cup \{\infty\}$ put $I_n := \{0, 1, 2, \dots, n\}$ if $n < \infty$ and $I_n := \mathbb{Z}^+$ if $n = \infty$. A function $\sigma : I_n \to N$ will be called a backward (partial) solution to f through x in N iff $\sigma(0) = x$ and

$$f(\sigma(i+1)) = \sigma(i) \text{ for all } i \in I_n \setminus \{0\}\,. \tag{4.1}$$

Note that the backward solution $\sigma : I_n \to N$ need not be unique and it may not exist at all.

Similarly, one defines $\tau : I_n \to N$ to be a forward (partial) solution to f through x in N iff $\tau(0) = x$ and

$$f(\tau(i)) = \tau(i+1) \text{ for all } i \in I_{n-1}\,. \tag{4.2}$$

Formula (4.2) implies that if τ is a forward solution to f through x, then $\tau(i) = f^i(x)$ for $i \in I_n$.

The family of all backward solutions to f through x in N defined on I_n will be denoted by $BS_n(x, N, f)$. Similarly, $FS_n(x, N, f)$ will stand for the family of forward solutions.

We also define functions

$$\alpha = \alpha_N = \alpha_{N,f} : N \ni x \to \sup\{n \in \mathbb{Z}^+ \mid BS_n(x, N, f) \neq \emptyset\}\,,$$

$$\omega = \omega_N = \omega_{N,f} : N \ni x \to \sup\{n \in \mathbb{Z}^+ \mid FS_n(x, N, f) \neq \emptyset\}\,,$$

and sets

$$\mathrm{Inv}^-(N, f) := \{x \in N \mid BS_\infty(x, N, f) \neq \emptyset\}\,,$$

$$\mathrm{Inv}^+(N,f) := \{x \in N \mid FS_\infty(x,N,f) \neq \emptyset\}\,.$$

We will omit f or N and f in the above notations if they are clear from context.

Assume now that N is an isolating neighborhood with respect to f.

Definition 4.1. (c.f. [Mr1, Def. 2.1]). The pair $P = (P_1, P_2)$ of compact subsets of N will be called a strong index pair iff

$$P_1, P_2 \text{ are positively invariant with respect to } N\,, \tag{4.3}$$

$$\mathrm{Inv}^- N \subset \mathrm{int}_N\, P_1, \qquad \mathrm{Inv}^+ N \subset N \backslash P_2\,, \tag{4.4}$$

$$P_1 \backslash P_2 \subset \mathrm{int}\, N \cap f^{-1}(\mathrm{int}\, N)\,. \tag{4.5}$$

Proposition 4.2. Each strong index pair is an index pair. QED

The family of strong index pairs in N will be denoted by $\mathrm{SIP}(N,f)$ or by $\mathrm{SIP}(N)$.

The proof of the following proposition is straightforward.

Proposition 4.3. Assume N is compact, $\{x_n\} \subset N$, $x_n \to x$ and for some $k \in \mathbb{N}$ and each $n \in \mathbb{N}$, $BS_k(x_n) \neq \emptyset (FS_k(x_n) \neq \emptyset)$. Then also $BS_k(x) \neq \emptyset$ $(FS_k(x) \neq \emptyset)$.

As an immediate consequence, we get

Proposition 4.4. α and ω are upper semi-continuous.

Lemma 4.5. *Assume N is compact and α_N or ω_N is unbounded. Then* $\mathrm{Inv}(N) \neq \emptyset$.

Proof: Assume α_N is unbounded. Then for each $n \in \mathbb{N}$ we can choose $x_n \in N$ such that $\alpha_N(x_n) = 2n$. Let $\sigma_n : I_{2n} \to N$ be the left solution through x_n in N. Put

$y_n := \sigma_n(n)$. Then, obviously $\alpha_N(y_n) \geq n$ and $\omega_N(y_n) \geq n$. Let y be a cluster point of $\{y_n\}$. Then, by Proposition 4.4, $\alpha_N(y) = \omega_N(y) = \infty$, i.e., $y \in \mathrm{Inv}(N)$.

If ω_N is unbounded, the proof is analogous. QED

We shall need the following m.v. map

$$\mathrm{Fd}_N := \mathrm{Fd}_{N,f} : N \ni x \to \{f^i(x) \mid i = 0, 1, \ldots, \omega_N(x)\} \subset N\,.$$

Like in [Mr1, Sect. 5] we prove the following two lemmas.

Lemma 4.6. *The m.v. mapping* Fd_N *is u.s.c. on* $N \setminus \mathrm{Inv}^+ N$.

Lemma 4.7. *Assume* Z *is compact and* $Z \cap \mathrm{Inv}^+ N = \emptyset$. *Then* $\mathrm{Fd}_N(Z)$ *is compact and* $\mathrm{Fd}_N(Z) \cap \mathrm{Inv}^+ N = \emptyset$, *i.e.* $N \setminus \mathrm{Fd}_N(Z)$ *is a neighborhood of* $\mathrm{Inv}^+ N$.

Lemma 4.8. *Assume* Z *is a compact neighborhood of* $\mathrm{Inv}^- N$ *in* N. *Then* $\mathrm{Fd}_N(Z)$ *is compact.*

Proof: Let $\{y_n\} \subset \mathrm{Fd}_N(Z)$, $y_n \to y \in N$. If $y \in \mathrm{Inv}^- N$ then $y \in Z \subset \mathrm{Fd}_N(Z)$. Otherwise $\alpha_N(y) < \infty$. Choose $\{j_n\} \subset \mathbb{Z}^+$, $\{z_n\} \subset Z$ such that $f^{j_n}(z_n) = y_n$, $j_n \in \{0, 1, \ldots, \omega_N(x)\}$. Now $\alpha_N(y) < \infty$ implies that $\{j_n\}$ is bounded. Hence, we can assume that $j_n \to j \in \mathbb{Z}^+$, $z_n \to z \in Z$. Thus $f^j(z) = y$ and $f^i(z) \in N$ for $i = 0, 1, \ldots, j$, i.e., $y \in \mathrm{Fd}_N(Z)$. It follows that $\mathrm{Fd}_N(Z)$ is compact. QED

Lemma 4.9. *Assume* A *is compact, positively invariant with respect to* N *and* $\mathrm{Inv}^- N \subset A$. *Then, for every open neighborhood* V *of* A, *there exists a compact neighborhood* Z *of* A *in* N *such that*

$$\mathrm{Fd}_N(Z) \subset V\,. \tag{4.6}$$

Proof: Let $k \in \mathbb{N}$ be such that $\alpha_N(y) < k$ for all $y \in N \setminus V$. For $x \in A$ put

$$n(x) := \min(k, \omega_N(x))\,.$$

Like in [Mr1, Lemma 5.8] find an open neighborhood Z of A such that

$$f^j(z) \in V \text{ for all } z \in Z \text{ and } j = 0, 1, \ldots, n(z) . \tag{4.7}$$

Take $x \in \mathrm{Fd}_N(Z)$. Then there exist $z \in Z_n$, $m \in \mathbb{Z}^+$ such that $x = f^m(z)$, $m \leq \omega_N(z)$. If $m \leq k$, then $m \leq n(z)$ and $x \in V$ by (4.7). If $m > k$ then $x \notin V$ implies $\alpha_N(x) < k$. But

$$\{f^{m-k}(z), f^{m-k+1}(z), \ldots, f^m(z)\} \subset N ,$$

$f^m(z) = x$ and $\alpha_N(x) \geq k$. Thus $x \in V$. QED

Theorem 1.2 is a straightforward consequence of the following lemma, which can be proved like in [Mr1] with lemmas 5.7, 5.8 replaced by lemmas 4.8, 4.9.

Lemma 4.10. *Assume U and V are open neighborhoods of $\mathrm{Inv}^+ N$ and $\mathrm{Inv}^- N$ respectively. Then there exists a strong index pair P in N such that*

$$P_1 \subset V, \quad N \backslash P_2 \subset U . \tag{4.8}$$

Definition 4.11. Assume $P, Q \in \mathrm{IP}(N)$. We will say that P is related to Q iff $P \subset Q$ and

$$Q_1 \backslash P_2 \subset f^{-1}(N) .$$

Like in [Mr1], we prove the following lemma.

Lemma 4.12. *For every isolating neighborhood N there exist $P, Q \in \mathrm{SIP}(N)$ such that $P \subset \mathrm{int}_N\, Q$ and P is related to Q. If $P, Q \in \mathrm{SIP}(N)$, $P \subset \mathrm{int}_N\, Q$ and P is related to Q, then there exists $R \in \mathrm{SIP}(N)$ such that $P \subset \mathrm{int}_N\, R$, $R \subset \mathrm{int}_N\, Q$, P is related to R, R is related to Q.*

5. The category of pairs and the quotient functor. It will be convenient to make the division process in P_1/P_2 functorial. For this end we introduce the category Prs, defined as follows. Its objects are pairs (P_1, P_2), where P_1 is a compact topological space and $P_2 \subset P_1$ is its closed subset. The set of morphisms from $P = (P_1, P_2)$ to $Q = (Q_1, Q_2)$ consists of all partial continuous maps $h : P_1 \multimap Q_1$ such that

$$\operatorname{dom} h \text{ is closed in } P_1 \,, \tag{5.1}$$

$$h(P_2) \subset Q_2 \,, \tag{5.2}$$

$$h(\operatorname{bd}_{P_1}(\operatorname{dom} h)) \subset Q_2 \,. \tag{5.3}$$

Observe that the identity map $\mathrm{id} : P_1 \to P_1$ satisfies the conditions (5.1-5.3) for any closed subset $P_2 \subset P_1$. We take it as the identity morphism of $P = (P_1, P_2)$ in Prs.

Assume $P, Q, R \in$ Prs and $\alpha : P \to Q$, $\beta : Q \to R$ are morphisms in Prs. We define the composition $\beta\alpha$ as the mapping

$$\beta\alpha : \alpha^{-1}(\operatorname{dom}\beta) \ni x \to \beta(\alpha(x)) \in R_1 \,.$$

Proposition 5.1. Prs constitutes a category.

Proof: The only thing we need to verify is whether the composition of morphisms in Prs is a morphism in Prs. Obviously $\beta\alpha$ is continuous and $\alpha^{-1}(\operatorname{dom}\beta)$, as closed in dom α, is also closed in P_1. Property (5.2) is trivial. To see that

$$\beta\alpha(\operatorname{bd}\alpha^{-1}(\operatorname{dom}\beta)) \subset R_2 \tag{5.4}$$

take $x \in \operatorname{bd}\alpha^{-1}(\operatorname{dom}\beta))$. Then $x \in \operatorname{dom}\alpha$ and $\alpha(x) \in \operatorname{dom}\beta$. If $x \in \operatorname{bd}(\operatorname{dom}\alpha)$ then $\alpha(x) \in Q_2$ and $(\beta\alpha)(x) \in R_2$, by (5.2). If $\alpha(x) \in \operatorname{bd}(\operatorname{dom}\beta)$ then $\beta(\alpha(x)) \in \beta(\operatorname{bd}(\operatorname{dom}\beta)) \subset R_2$. Otherwise, $x \in \operatorname{int}(\operatorname{dom}\alpha)$ and $\alpha(x) \in \operatorname{int}(\operatorname{dom}\beta)$. But this contradicts $x \in \operatorname{bd}(\alpha^{-1}(\operatorname{dom}\beta))$. Hence (5.3) is proved and $\beta\alpha$ is a morphism in Prs. QED

With every object $P = (P_1, P_2) \in \mathrm{Prs}$ we associate the quotient space P_1/P_2 defined by

$$P_1/P_2 := P_1 \backslash P_2 \cup \{[P_2]\}$$

and endowed with the strongest topology for which the projection $q_P : P_1 \to P_1/P_2$ given by

$$q_P(x) := \begin{cases} x & \text{if } x \in P_1 \backslash P_2, \\ P_2 & \text{otherwise} \end{cases}$$

is continuous.

Assume $P, Q \in \mathrm{Prs}$ and $\alpha : P \to Q$ is a morphism in Prs. Put

$$\mathrm{Quot}(P) := (P_1/P_2, P_2), \tag{5.5}$$

$$\mathrm{Quot}(\alpha)(x) := \begin{cases} q_Q(\alpha(x)) & \text{if } x \neq [P_2] \text{ and } x \in \ \mathrm{dom}\ \alpha, \\ [Q_2] & \text{otherwise.} \end{cases} \tag{5.6}$$

Proposition 5.2. Formula (5.6) defines a continuous map

$$\mathrm{Quot}(\alpha) : P_1/P_2 \to Q_1/Q_2 \, .$$

Proof: Put $\alpha' := \mathrm{Quot}(\alpha)$ and take $x \in P_1/P_2$. Let U' be a neighborhood of $\alpha'(x)$. Put $U := q_{Q^{-1}}(U')$. Then $U \supset Q_2$ if $\alpha'(x) = [Q_2]$. We can also assume that $U \cap Q_2 = \emptyset$, i.e. $U = U'$, if $\alpha'(x) \neq [Q_2]$. Since $\alpha : \mathrm{dom}\, \alpha \to Q_1$ is continuous, we will find V open in P_1, such that $V \cap \mathrm{dom}\, \alpha = \alpha^{-1}(U)$. We will consider two cases. First assume that $\alpha'(x) = [Q_2]$. Then, by (5.2),

$$P_2 = (P_2 \cap \mathrm{dom}\ \alpha) \cup (P_2 \backslash \mathrm{dom}\ \alpha) \subset \alpha^{-1}(Q_2) \cup (P_2 \backslash \mathrm{dom}\ \alpha) \subset$$
$$\alpha^{-1}(U) \cup (P_1 \backslash \mathrm{dom}\ \alpha) \subset V \cup (P_1 \backslash \mathrm{dom}\ \alpha) =: W \, .$$

Put $W' := q_P(W)$. Then W' is an open neighborhood of x in P_1/P_2 and $\alpha'(W') \subset U'$.

Hence it remains to consider the case $\alpha'(x) \neq [Q_2]$. Then, by (5.3), $x \in \mathrm{int}(\mathrm{dom}\ \alpha)$. Thus we can find a set $V_1 \subset V$ open in P_1 such that $x \in V_1 \subset \mathrm{int}(\mathrm{dom}\ \alpha)$. It follows that

$$\alpha'(V_1) = \alpha(V_1) = \alpha(V_1 \cap \mathrm{dom}\ \alpha) \subset \alpha(V \cap \mathrm{dom}\ \alpha) \subset U = U' \, .$$

QED

Once we know $\mathrm{Quot}(\alpha)$ is continuous, it is a straightforward task to verify the following.

Corollary 5.3. Formulae (5.5-5.6) define a covariant functor $\mathrm{Quot} : \mathrm{Prs} \to \mathrm{Comp}_*$.

Proposition 5.4. Assume $\alpha : P \to P$ is a morphism in Prs. Then $\mathrm{Quot}(\alpha) = \mathrm{id}$ iff the following two conditions are satisfied.

$$P_1 \backslash P_2 \subset \mathrm{dom}\ \alpha \tag{5.7}$$

$$\alpha(x) = x \text{ for } x \in P_1 \backslash P_2\,. \tag{5.8}$$

Proof: Put $\alpha' := \mathrm{Quot}(\alpha)$ and assume $\alpha' = \mathrm{id}$. It follows that $x \neq [P_2] \Rightarrow \alpha'(x) \neq \alpha'([P_2]) = [P_2]$. Since $\alpha'(x) = [P_2]$ for $x \notin \mathrm{dom}\ \alpha$, we get $P_1 \backslash P_2 \subset \mathrm{dom}\ \alpha$. Moreover, if $x \in P_1 \backslash P_2$ then $\alpha(x) = \alpha'(x) = x$. Hence (5.7), (5.8) are proved.

If (5.7), (5.8) are satisfied, we have for $x \in P_1 \backslash P_2$, $\alpha'(x) = \alpha(x) = x$. Obviously also $\alpha'(P_2) = [P_2]$, hence $\alpha' = \mathrm{id}$. QED

Theorem 5.5 (excision property for the functor Quot**).** *Assume $P, Q \in \mathrm{Prs}$ are such that*

$$P_1 \backslash P_2 = Q_1 \backslash Q_2 \textit{ and } P_1 \textit{ is closed in } Q_1\,. \tag{5.9}$$

Then $\mathrm{Quot}(i_{P,Q})$ *is an isomorphism in* Comp_*.

Proof: We will show that $j := \mathrm{id}_P$ is a morphism from Q to P in Prs. Properties (5.1-5.2) are obviously satisfied. We have also

$$\mathrm{bd}(\mathrm{dom}\ j) = \mathrm{bd}\ P_1 \subset \mathrm{cl}(Q_1 \backslash P_1) \subset Q_2\,,$$

because, by (5.9), $Q_1 \backslash P_1 \subset Q_2$ and Q_2 is closed in Q_1. This shows (5.3).

Now $j \circ i_{P,Q} = \mathrm{id}_P$, the identity on P in Prs, hence $\mathrm{Quot}(j \circ i_{P,Q})$ is an identity in Comp_*. By proposition 5.4, $\mathrm{Quot}(i_{P,Q} \circ j)$ is also an identity in Comp_*. It follows that $\mathrm{Quot}(i_{P,Q})$ and $\mathrm{Quot}(j)$ are mutually inverse. QED

A morphism $H \in \mathrm{Prs}(P \times I, Q)$ with $I = [0,1]$ will be called a homotopy from P to Q in Prs. For each $t \in I$ one defines the partial morphism $H_t \in \mathrm{Prs}(P, Q)$ given by

$$H_t := H \circ i_t \,,$$

where $i_t \in \mathrm{Prs}(P, P \times I)$ is the full morphism

$$i_t := P \ni x \to (x,t) \in P \times I \,.$$

We say that two morphisms $f, g \in \mathrm{Prs}(P,Q)$ are homotopic if there exists a homotopy $H \in \mathrm{Prs}(P \times I, Q)$ such that $f = H_0$, $g = H_1$. We write then $f \sim g$ in Prs.

The following proposition can be verified as in the case of usual homotopies.

Proposition 5.6. The homotopy between morphisms in Prs is an equivalence relation, which coincides with the composition of morphisms.

Thus we can introduce the homotopy category of pairs, which has the same objects as Prs and equivalence classes of morphisms homotopic in Prs as morphisms. We will denote this category by HPrs.

Proposition 5.7. If f and g are homotopic in Prs then $\mathrm{Quot}(f)$ and $\mathrm{Quot}(g)$ are homotopic in Comp_*.

Proof: Observe that

$$(P_1/P_2) \times I \backslash [P_2] \times I = P_1 \times I / P_2 \times I \backslash [P_2 \times I] \,.$$

Consider the map

$$\kappa : (P_1/P_2) \times I \to P_1 \times I / P_2 \times I$$

which is identity on $(P_1/P_2) \times I \backslash [P_2] \times I$ and sends all points in $[P_2] \times I$ to $[P_2 \times I]$. It is straightforward to verify that κ is continuous. Let H be a homotopy in Prs such that $H_0 = f$ and $H_1 = g$. Put

$$H' := \mathrm{Quot}(H) \circ \kappa\,.$$

One can easily verify that H' is a homotopy in Comp_* joining $\mathrm{Quot}(f)$ and $\mathrm{Quot}(g)$. QED

Hence we can also consider the functor Quot as a functor Quot : HPrs $\to$ H$Comp_*$.

Assume $f : X \to X$ is a continuous map and $P, Q \in$ Prs consist of subsets of X. Define the map

$$f_{PQ} : P_1 \cap f^{-1}(Q_1) \ni x \in f(x) \in Q_1\,.$$

Note that this map need not be a morphism from P to Q in Prs. However, we have the following.

Proposition 5.8. If f satisfies

$$P_1 \backslash P_2 \subset f^{-1}(Q_1) \tag{5.10}$$

and

$$P_2 \cap f^{-1}(Q_1) \subset f^{-1}(Q_2) \tag{5.11}$$

then $f_{PQ} \in \mathrm{Prs}(P, Q)$.

Proof: Since dom $f_{PQ} = P \cap f^{-1}(Q)$ is closed in P, we need to verify only properties (5.2), (5.3). By (5.11), there is

$$f_{PQ}(P_2) = f(P_2 \cap f^{-1}(Q_1)) \subset f(f^{-1}(Q_2)) \subset Q_2\,,$$

which proves (5.2). By (5.10) we have $\mathrm{cl}_{P_1}(P_1 \backslash f^{-1}(Q_1)) \subset P_2$, hence, by (5.11)

$$\mathrm{bd}_{P_1} P_1 \cap f^{-1}(Q_1) = P_1 \cap f^{-1}(Q_1) \cap \mathrm{cl}_{P_1}(P_1 \backslash f^{-1}(Q_1)) \subset P_2 \cap f^{-1}(Q_1) \subset f^{-1}(Q_2)$$

which proves (5.3). QED

6. The construction of the Conley index.

It follows from Proposition 5.8 that $f_{PP} \in \mathrm{Prs}(P,P)$. We give the following definition.

Definition 6.1. Both f_{PP} and $f_P := \mathrm{Quot}(f_{PP}) : P_1/P_2 \to P_1/P_2$ will be called index maps associated with the index pair P (and the map f). This will cause no misunderstanding.

Every index pair P together with the associated index map f_{PP} constitutes an object $P_f := (P, f_{PP}) \in \mathrm{Endo}(\mathrm{Prs})$. If we put $T' := T \circ \mathrm{Quot}$, Theorem 1.7 can be reformulated as follows.

Theorem 6.2′. *Assume K is an isolated invariant set with respect to $f : X \multimap X$. Then $LT'(P_f)$ and $LT'(Q_f)$ are isomorphic objects in $\mathrm{Auto}(\mathcal{E})$ for all isolating neighborhoods N, M of K and $P \in \mathrm{IP}(N)$, $Q \in \mathrm{IP}(M)$.*

Obviously, in order to prove Theorem 1.7, it suffices to prove the above theorem. Theorem 6.2 can be proved as in [Mr1, Th. 2.6] if we are able to prove the following.

Theorem 6.3. *Assume $f : X \multimap X$ is a map, N is an isolating neighborhood with respect to f and $P, Q \in \mathrm{IP}(N)$ are such that $P \subset Q$. If $\iota : P \to Q$ denotes the inclusion and $T' = T \circ \mathrm{Quot}$ then $T'(\iota) : P_f \to Q_f$ is a morphism in $\mathrm{Endo}(\mathcal{E})$ and $LT'(\iota)$ is an isomorphism in $\mathrm{Auto}(\mathcal{E})$.*

Proof: We assume that T is a covariant functor. The proof for a contravariant functor is similar. First we assume that

$$P \text{ is related to } Q\,, \tag{6.1}$$

$$f(Q) \cap N \subset P\,. \tag{6.2}$$

Condition (6.2) and Proposition 5.9 imply that $f_{QP} \in \mathrm{Prs}(Q, P)$. Hence we have the following commutative diagram, in which vertical arrows denote inclusions

$$
\begin{array}{ccc}
 & f_{PP} & \\
P & \longrightarrow & P \\
\iota \downarrow & \nearrow f_{QP} & \downarrow \iota \\
Q & \longrightarrow & Q \\
 & f_{QQ} &
\end{array}
$$

Applying T' to the above diagram, we obtain the following commutative diagram

$$
\begin{array}{ccc}
 & T'(f_P) & \\
T'(P) & \longrightarrow & T'(P) \\
T'(\iota) \Big\downarrow & \nearrow T'(f_{QP}) & \Big\downarrow T'(\iota) \\
T'(Q) & \longrightarrow & T'(Q) \\
 & T'(f_Q) &
\end{array}
$$

Now Theorem 1.4 shows that $LT'(\iota)$ is an isomorphism.

The rest of the proof proceeds as in Steps 2, 3 of the proof of Theorem 6.4 in [Mr1] with the difference that we use Theorem 5.6 instead of the strong excision property of the Alexander-Spanier cohomology.

7. Proof of the homotopy property. In this and the following section T' denotes $T \circ \mathrm{Quot}$. Assume $\Lambda \subset \mathbb{R}$ is a compact interval, $U \subset X$ is open. For $\Delta \subset \Lambda$ consider the mapping

$$f(\Delta) : \Delta \times U \ni (\lambda, x) \to (\lambda, f(\lambda, x)) \in \Delta \times X$$

defined for $(\lambda, x) \in \mathrm{dom}\, f$ such that $\lambda \in \Lambda$.

We will simply write f_λ instead of $f(\{\lambda\})$ and λ or Δ instead of f_λ or $f(\Delta)$ in all cases, where f_λ or $f(\Delta)$ appear as a parameter.

Proposition 7.1. Assume $N \subset U$ is compact. Then the m.v. mappings

$$\Lambda \ni \lambda \to \mathrm{Inv}^-(N, \lambda) \subset N, \qquad \Lambda \ni \lambda \to \mathrm{Inv}^+(N, \lambda) \subset N$$

are u.s.c.

Corollary 7.2. If for some $\mu \in \Lambda$, N is an isolating neighborhood with respect to f_μ, then N is an isolating neighborhood with respect to f_λ for λ sufficiently close to μ.

Lemma 7.3. *Assume $I \subset \mathbb{R}$ is an interval, $\mu \in I$, $F : X \times I \longrightarrow\!\!\!\!\rightarrow Y$ is a m.v. map, U is open in Y and $A \subset X$ is compact such that for every $x \in A$ there exist a neighborhood V of x in X and a neighborhood Δ of μ in I such that $F(V \times \Delta) \subset U$. Then there exist μ', $\mu'' \in I$ such that $\mu \in (\mu', \mu'')$ and $F(A \times (\mu', \mu'')) \subset U$.*

Proof. Compactness. QED

Lemma 7.4. *If $P, Q \in \mathrm{SIP}(N, \mu)$ are such that $P \subset \mathrm{int}_N Q$ and P is related to Q then there exists Λ_0 a neighborhood of μ in Λ such that for every interval $\Delta \subset \Lambda_0$, $P(\Delta) :=$ $\mathrm{Fd}_{N\times\Delta,\Delta}(P\times\Delta)$ is an index pair in $N\times\Delta$ with respect to $f(\Delta)$ and $P\times\Delta \subset P(\Delta) \subset Q\times\Delta$.*

Proof: By (5.2) and Proposition 7.1 we have

$$\mathrm{Inv}^-(N, \Delta) \subset \mathrm{int}_{N\times\Delta}(P_1 \times \Delta) \tag{7.1}$$

$$\mathrm{Inv}^+(N, \Delta) \subset (N \backslash Q_2) \times \Delta \tag{7.2}$$

for Δ contained in some sufficiently small neighborhood Λ_1 of μ. Upper semicontinuity of Fd and continuity of f allow us to apply Lemma 7.3 three times respectively to

$$F = \mathrm{Fd}_{N\times\Lambda_1,\Lambda_1}, \quad U = \mathrm{int}_N Q_1 \times \Lambda_1, A = P_1,$$

$$F = \mathrm{Fd}_{N\times\Lambda_1,\Lambda_1}, \quad U = \mathrm{int}_N Q_2 \times \Lambda_1, A = P_2,$$

$$F = f(\Lambda_1), \qquad U = \mathrm{int}\, N \times \Lambda_1, A = \mathrm{cl}(Q_1 \backslash P_2).$$

Hence we can find $\Lambda_0 \subset \Lambda$ such that

$$\mathrm{Fd}_{N\times\Lambda_1,\Lambda_1}(P_1 \times \Lambda_0) \subset \mathrm{int}_N Q_i \times \Lambda_0 \text{ for } i = 1, 2 \tag{7.3}$$

$$f(\mathrm{cl}(Q_1 \backslash P_2) \times \Lambda_0) \subset \mathrm{int}\, N\,. \tag{7.4}$$

In particular, by (7.3), for every compact interval $\Delta \subset \Lambda_1$

$$P_i \times \Delta \subset P_i(\Delta) \subset Q_i \times \Delta\,. \tag{7.5}$$

Compactness of $P_1(\Delta)$ follows from (7.1) and Lemma 4.8. Compactness of $P_2(\Delta)$ follows from (7.2) and Lemma 4.7. Positive invariance with respect to $N \times \Delta$ is obvious. Finally, by (7.1-7.3)

$$\mathrm{Inv}(N, \Delta) = \mathrm{Inv}^-(N, \Delta) \cap \mathrm{Inv}^+(N, \Delta) \subset \mathrm{int}_{N\times\Delta}(P_1(\Delta)\backslash P_2(\Delta))$$

and by (7.5) and (7.4)

$$P_1(\Delta)\backslash P_2(\Delta) \subset (Q_1 \backslash P_2) \times \Delta \subset f(\Delta)^{-1}(N) \times \Delta\,.$$

QED

Lemma 7.5. *Assume $N \subset X$, $\mu \in \Lambda$, $P, Q \in \mathrm{SIP}(N, \mu)$ are such as in the previous lemma. Then there exists Λ_0, a neighborhood of μ in Λ, such that for every $\kappa \in \Lambda_0$ there exists $R(\kappa) \in \mathrm{IP}(N, \kappa)$ satisfying $P \subset R(\kappa) \subset Q$ and such that the inclusions*

$$i : P \to R(\kappa), \quad j : R(\kappa) \to Q$$

induce morphisms

$$T'(i) : T'(P_{f\mu}) \to T'(R(\kappa)_{f_\kappa})\,,$$

$$T'(j) : T'(R(\kappa)_{f_\kappa}) \to T'(Q_{f_\mu})$$

in $\mathrm{Endo}(\mathcal{E})$*. (This applies to a covariant functor T. Arrows are reversed if T is contravariant.)*

Proof: We assume that T is covariant, the proof for a contravariant functor being similar. Applying Lemma 4.12 find R, $S \in \mathrm{SIP}(N)$ such that $P \subset \mathrm{int}_N\, R \subset R \subset \mathrm{int}_N\, S \subset$

$S \subset \operatorname{int}_N Q$. It follows from Lemma 7.4 that we can find a neighborhood Λ_0 of μ such that for every compact interval $\Delta \subset \Lambda_0$ there exist index pairs $P(\Delta)$, $R(\Delta)$, $S(\Delta)$ satisfying

$$P \times \Delta \subset P(\Delta) \subset R \times \Delta \subset R(\Delta) \subset S \times \Delta \subset S(\Delta) \subset Q \times \Delta\,.$$

Fix $\kappa \in \Delta_0$. Let I denote the closed interval with endpoints μ, κ. Put

$$H := f_{P\times I,R(\kappa)}, \quad G := f_{R(\kappa)\times I,Q}\,.$$

We will show that $H \in \operatorname{Prs}(P \times I, R(\kappa))$, $G \in \operatorname{Prs}(R(\kappa) \times I, Q)$. Since obviously $\operatorname{dom} H$ is closed, we need to prove only properties (5.2)-(5.3). Assume $(x,\lambda) \in (P_2 \times I) \cap f^{-1}(R_1(\kappa))$. Then

$$x \in P_2 \cap f_{\lambda^{-1}}(R_1(\kappa) \subset P_2(\lambda) \cap f_{\lambda^{-1}}(R_1(\kappa) \subset P_2(\lambda) \cap f_{\lambda^{-1}}(N) \subset$$
$$f_{\lambda^{-1}}(P_2(\lambda)) \subset f_{\lambda^{-1}}(R_2) \subset f_{\lambda^{-1}}(R_2(\kappa))\,,$$

i.e., $P_2 \times I \cap f^{-1}(R_1(\kappa)) \subset f^{-1}(R_2(\kappa))$, which proves (5.2).

We have

$$P_1 \times I \cap f^{-1}(R_1(\kappa)) \cap \operatorname{cl}(P_1 \times I \backslash f^{-1}(R_1(\kappa)) \subset$$
$$\subset P_1(I) \cap f(I)^{-1}(N \times I) \cap \operatorname{cl}(P_1(I) \backslash f(I)^{-1}(P_1(I))) \subset$$
$$\subset \operatorname{bd}\ \operatorname{dom} f_{P(I)} \subset f_{P(I)^{-1}}(P_2(I)) \subset f^{-1}(R_2(\kappa))\,.$$

This proves that $H \in \operatorname{Prs}(P \times I, R(\kappa))$. The proof that $G \in \operatorname{Prs}(R(\kappa) \times I, Q)$ is analogous.

It is now straightforward to verify that H is a homotopy joining the maps $i_{P,R(\kappa)} \circ f_{\mu,P}$ and $f_{\kappa,R(\kappa)} \circ i_{P,R(\kappa)}$ and G is a homotopy joining the maps $i_{R(\kappa),Q} \circ f_{\kappa,R(\kappa)}$ and $f_{\mu,Q} \circ i_{R(\kappa),Q}$. Hence, applying T', we get the following commutative diagram in $\mathcal{E}$.

$$\begin{array}{ccc}
T'(P) & \xrightarrow{T'(f_{\mu,P})} & T'(P) \\
\downarrow & & \downarrow \\
T'(R(\kappa)) & \xrightarrow{T'(f_{\kappa,R(\kappa)})} & T'(R(\kappa)) \\
\downarrow & & \downarrow \\
T'(Q) & \xrightarrow{T'(f_{\mu,Q})} & T'(Q)
\end{array}$$

with vertical arrows induced by inclusions. But this proves our assertion. QED

Proof of Theorem 1.10. Obviously it is sufficient to show that for any $\mu \in \Lambda$ there exists Λ_0 a neighborhood of μ such that for all $\nu \in \Lambda_0$

$$C(\mathrm{Inv}(N,\mu)) = C(\mathrm{Inv}(N,\nu)).$$

Thus fix $\mu \in \Lambda$ and using Lemma 4.12 find index pairs $P, Q, R \in \mathrm{IP}(N,\mu)$ such that $P \subset \mathrm{int}_N Q$, $Q \subset \mathrm{int}_N R$, P is related to Q, Q is related to R. Applying Lemma 7.5 twice, we can find Λ_0 a neighborhood of μ in Λ such that for every $\lambda \in \Lambda_0$ there exist index pairs $P(\lambda)$, $Q(\lambda) \in \mathrm{IP}(N,\lambda)$ satisfying $P \subset P(\lambda) \subset Q \subset (Q(\lambda)$ and such that we have the following commutative diagram of maps induced by inclusions

$$\begin{array}{ccc}
T'(P(\lambda)_{f_\lambda}) & \xrightarrow{T'(j_1)} & T'(Q_{f_\mu}) \\
{\scriptstyle T'(j_0)}\uparrow & & \downarrow{\scriptstyle T'(j_2)} \\
T'(P_{f_\mu}) & \xrightarrow[T'(j)]{} & T'(Q(\lambda)_{f_\lambda})
\end{array}$$

By applying functor L to the above diagram, we get from Theorem 6.3 that $LT'(j_1) \circ LT'(j_0) = LT'(j_1 \circ j_0)$, $LT'(j_2) \circ LT'(j_1) = LT'(j_2 \circ j_1)$ are isomorphisms, thus $LT'(j)$ is also an isomorphism. Hence

$$C(\mathrm{Inv}(N,\mu)) = LT'(P) = LT'(Q(\lambda)) = C(\mathrm{Inv}(N,\lambda)),$$

which finishes the proof. QED

8. Proofs of additivity and commutativity properties.

Proof of Theorem 1.11. Choose U_1, U_2 to be disjoint, open neighborhoods of K_1 and K_2 respectively. For $i = 1, 2$ let N_i be a compact neighborhood of K_i such that

$$N_i \subset U. \tag{8.1}$$

Select P', P'' index pairs of K_1 and K_2 in N_1 and N_2 respectively. One can easily verify that $P := P' \cup P''$ is an index pair of K in $N := N_1 \cup N_2$. Obviously $f_P = f_{P'} \cup f_{P''}$.

Hence, since T preserves coproducts, $T'(P_f)$ is the coproduct of $T'(P'_f)$ and $T'(P''_f)$. It suffices now to apply L. QED

Proof of Theorem 1.12. Choose an isolating neighborhood M such that $\operatorname{Inv} M = K$. Since $\varphi(K)$ is compact and by the invariance of K with respect to f we get $\psi(\varphi(K)) = f(K) = K$. Hence we can find a compact neighborhood N of $\varphi(K)$ such that

$$\psi(N) \subset \operatorname{int} M\,. \tag{8.2}$$

We will show that $\operatorname{Inv}(N, g) = \varphi(K)$. Note that $g(\varphi(K)) = \varphi\psi\varphi(K) = \varphi(f(K)) = \varphi(K)$. Hence $\varphi(K)$ is g-invariant, i.e., $\varphi(K) = \operatorname{Inv}(\varphi(K), g) \subset \operatorname{Inv}(N, g)$. To show the opposite inclusion, take $y \in \operatorname{Inv}(N, g)$. Let $\tau : \mathbb{Z} \to N$ be a solution to g through y. Put $\sigma := \psi\tau$. It is straightforward to verify that σ is a solution to f through $\psi(y)$. Moreover, (8.2) implies that σ is a solution in M. In particular $\sigma(-1) \in \operatorname{Inv} M = K$, hence

$$y = \tau(0) = g(\tau(-1)) = \varphi(\sigma(-1)) \in \varphi(K)\,.$$

This proves that $\varphi(K)$ is an isolated invariant set and N isolates $\varphi(K)$.

Put $M' := M \cap \varphi^{-1}(N)$. Observe that $\operatorname{Inv} M' \subset \operatorname{Inv} M = K$ and also $K \subset \varphi^{-1}(N)$, which implies $\operatorname{Inv} M' = K$. Moreover,

$$K \subset \operatorname{int} M \cap \operatorname{int} \varphi^{-1}(N) = \operatorname{int} M'\,.$$

Thus M' is an isolating neighborhood for K.

Select $Q \in \operatorname{IP}(N)$ and put $P_i := M' \cap \varphi^{-1}(Q_i)$ for $i = 1, 2$. We will show that $P := (P_1, P_2) \in \operatorname{IP}(M')$. First observe that if $x \in P_i$ $(i = 1, 2)$ and $f(x) \in M'$ then $\varphi(x) \in Q_i$, $g(\varphi(x)) = \varphi(f(x)) \in N$. Since $Q \in \operatorname{IP}(N)$, we get $\varphi(f(x)) \in Q_i$, i.e. $f(x) \in P_i$. This proves (1.1). Now take $x \in P_1 \backslash P_2$. Then $\varphi(x) \in Q_1 \backslash Q_2 \subset G^{-1}(N)$, i.e. $\varphi f(x) = g\varphi(x) \in N$. Thus $f(x) \in \varphi^{-1}(N)$. Moreover, $f(x) = \psi\varphi(x) \in \psi(Q_1) \subset \psi(N) \subset M$ and we get $f(x) \in M$, which proves (1.2). Finally, since $\varphi(K) \subset \operatorname{int}(P_1 \backslash P_2)$, we can find an open neighborhood $U \subset \operatorname{int} M$ of K such that $\varphi(U) \subset \operatorname{int}(Q_1 \backslash Q_2)$. Then $U \subset \operatorname{int}(\varphi^{-1}(Q_1) \backslash \varphi^{-1}(Q_2))$ and

$$K \subset U \subset \operatorname{int} M \cap \operatorname{int}(\varphi^{-1}(Q_1) \backslash \varphi^{-1}(Q_2)) = \operatorname{int}(P_1 \backslash P_2)\,.$$

This proves that $P \in \mathrm{IP}(M')$.

One can easily verify the assumptions of Proposition 5.8 to see that $\varphi_{PQ} \in \operatorname{Prs}(P,Q)$, $\psi_{QP} \in \operatorname{Prs}(Q,P)$. Moreover, $\psi_{QP} \circ \varphi_{PQ} = f_P$, $\varphi_{PQ} \circ \psi_{QP} = g_Q$, hence, we have a commutative diagram in $\mathcal{E}$ (if T is contravariant, then the arrows are reversed):

$$\begin{tikzcd}
T'(P) \arrow[r, "T(f_P)"] \arrow[d, "T'(\varphi_{PQ})"'] & T'(P) \arrow[d, "T'(\varphi_{PQ})"] \\
T'(Q) \arrow[r, "T(g_Q)"'] \arrow[ur, "T'(f_{QP})"] & T'(Q)
\end{tikzcd}$$

It follows now from Theorem 1.4 that $LT'(\varphi_{PQ})$ is an isomorphism in $\operatorname{Auto}(\mathcal{E})$, i.e., $C(K,f) = LT'(P_f)$ and $C(\varphi(K),g) = LT'(Q_g)$ are isomorphic. QED

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