Singular Index Pairs

Konstantin Mischaikow*
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160
mischaik@math.gatech.edu

Marian Mrozek †
Computer Science Department
Jagiellonian University
Kraków, Poland

James F. Reineck‡
Department of Mathematics
SUNY at Buffalo
Buffalo, NY 14214

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1 Introduction

Following N. Fenichel's paper [4] the ideas of dynamical systems have been applied to problems involving singular perturbations resulting in a technique often referred to as the theory of geometric singular perturbations (see also the earlier works on invariant manifolds [5, 3]). The fundamental concepts involve invariant manifolds, hyperbolicity, and transversality. The usefulness of these methods can be readily seen through the works of a growing number of authors (see [6] and the references therein). The purpose of this paper is to suggest the possibility of a topological theory of singular perturbations based on isolating neighborhoods. The foundations of our work can be found in a little known or at least seldom cited paper of C. Conley [2]. In it Conley introduces the idea of a singular isolating neighborhood, i.e. a compact set which is not an isolating neighborhood for a particular parameter value, but does isolate for nearby flows. Perhaps one of the reasons that this idea has not been exploited is that isolating neighborhoods alone provide little useful information about the dynamics. For this reason most work involving isolating neighborhoods also involves the Conley index (see [1, 12, 11]). The goal of this paper is to extend the Conley index to the setting of singular flows, i.e. for flows in which there is not an isolating neighborhood.

To be more precise, consider the family of differential equations on $\mathbf{R}^n$ given by

$$\dot{x} = f(x,\epsilon) = f_0(x) + \sum_{i=1}^{m} \epsilon^i f_i(x) + o(\epsilon^m) \tag{1}$$

at $\epsilon = 0$ where $\epsilon \geq 0$. For each ϵ let $\varphi^\epsilon : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ denote the flow generated by (1) with parameter value ϵ.

Recall that given a set $N \subset \mathbf{R}^n$, its *maximal invariant set* under the flow φ^ϵ is defined by

$$\mathrm{Inv}(N, \varphi^\epsilon) := \{x \in N \mid \varphi^\epsilon(\mathbf{R}, x) \subset N\},$$

and that a compact set N is an *isolating neighborhood* if

$$\mathrm{Inv}(N, \varphi^\epsilon) \subset \mathrm{int} N.$$

It is well known that if N is an isolating neighborhood for φ^0 then there exists $\bar{\epsilon} > 0$ such that for all $\epsilon \in [0, \bar{\epsilon}]$, N is an isolating neighborhood for φ^ϵ. The theory we are interested in developing begins with the assumption that N is *not* an isolating neighborhood for φ^0.

Definition 1.1 A compact set $N \subset \mathbf{R}^n$ is called a *singular isolating neighborhood* if N is not an isolating neighborhood for φ^0, but there is an $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon}]$, N is an isolating neighborhood for φ^ϵ.

As was mentioned earlier one of the few tools for analyzing the dynamics of the invariants sets within an isolating neighborhood is the Conley index (see [9] for a brief survey of such techniques). Following the standard assumption underlying the technique of singular perturbations we assume that the dynamics of the "singular" system $\dot{x} = f_0(x)$ is simpler to analyze than that of the perturbed system. Thus, our goal is to determine Conley indices for isolated invariant sets for small $\epsilon > 0$ from the dynamics of φ^0.

If N is an isolating neighborhood for φ^0, then one can find an index pair (detailed definitions and references will be given later) (N', L) for $\mathrm{Inv}(N, \varphi^0)$ such that the cohomological Conley index is given by

$$CH^*(\mathrm{Inv}(N, \varphi^0)) := H^*(N', L).$$

Furthermore, since the Conley index is stable under perturbation it follows that there exists $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon}]$,

$$CH^*(\mathrm{Inv}(N, \varphi^\epsilon)) \approx CH^*(\mathrm{Inv}(N, \varphi^0)).$$

This motivates the following definition.

Definition 1.2 A pair of compact sets (N, L) with $L \subset N$ is a *singular index pair* if $\mathrm{cl}(N \setminus L)$ is a singular isolating neighborhood and there exists $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon}]$

$$H^*(N, L) \approx CH^*(\mathrm{Inv}(\mathrm{cl}(N \setminus L), \varphi^\epsilon)).$$

Remark 1.3 Throughout this paper H^* denotes Alexander–Spanier cohomology [7, 13].

By themselves these definitions are best described as wishful thinking. The content of this paper is Theorem 1.16 which given an appropriate singular isolating neighborhood N dictates a method for constructing a set L such that (N, L) is a singular index pair. We now turn to a description of these results.

Observe that if a compact set N is not an isolating neighborhood under φ^0, then there exists $x \in \mathrm{Inv}(N, \varphi^0) \cap \partial N$. If, in addition, N is a singular isolating neighborhood then for $\epsilon > 0$, but sufficiently small, $x \notin \mathrm{Inv}(N, \varphi^\epsilon)$ and hence x must leave N in either forward or backward time. This suggests the following definition.

Definition 1.4 Let N be a compact set and let $x \in \mathrm{Inv}(N, \varphi^0)$. x is a *slow exit (entrance) point* if there exists a neighborhood U of x and an $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon}]$ there exists a time $T(\epsilon, U) > 0$ ($T(\epsilon, U) < 0$) satisfying

$$\varphi^\epsilon(T(\epsilon, U), U) \cap N = \emptyset.$$

Observe that U is independent of ϵ and since $x \in \mathrm{Inv}(N, \varphi^0)$,

$$\lim_{\epsilon \to 0} T(\epsilon, U) = \infty.$$

This definition makes the following theorem a triviality.

Theorem 1.5 *Let N be a compact set. If $\mathrm{Inv}(N, \varphi^0) \cap \partial N$ consists of slow exit and slow entrance points, then N is a singular isolating neighborhood.*

Proof: It needs to be shown that there exists $\bar{\epsilon} > 0$ such that if $x \in \partial N$ and $\epsilon \in (0, \bar{\epsilon}]$, then $\varphi^\epsilon(\mathbf{R}, x) \not\subset N$. Let $x \in \mathrm{Inv}(N, \varphi^0) \cap \partial N$. Then there exists U_x a neighborhood of x, $\epsilon_x > 0$, and $T(\epsilon, U_x)$ such that

$$\varphi^\epsilon(T(\epsilon, U_x), U_x) \cap N = \emptyset$$

for all $\epsilon \in (0, \epsilon_x]$. Since $\mathrm{Inv}(N, \varphi^0) \cap \partial N$ is compact we can cover it with a finite number of U_x's, i.e. $U_{x_1}, \ldots, U_{x_k}$. Let

$$U = \cup_{i=1}^k U_{x_i}, \;\; T(\epsilon) = \max_{i=1,\ldots k} |T(\epsilon, U_{x_i})| \text{ and } \bar{\epsilon}_1 = \min_{i=1,\ldots k} \epsilon_{x_i}.$$

Then for every $x \in U$ and $\epsilon \in (0, \bar{\epsilon}_1]$,

$$\varphi^\epsilon([-T(\epsilon), T(\epsilon)], x) \not\subset N.$$

On the other hand, $\partial N \setminus U$ is compact and for all $x \in \partial N \setminus U$, $\varphi^0(\mathbf{R}, x) \not\subset N$. Therefore, by uniform continuity there exists $\bar{\epsilon}_2 > 0$ such that for all $\epsilon \in (0, \bar{\epsilon}_2]$ and for all $x \in \partial N \setminus U$, $\varphi^\epsilon(\mathbf{R}, x) \not\subset N$. Let $\bar{\epsilon} = \min\{\bar{\epsilon}_1, \bar{\epsilon}_2\}$. ∎

The proof of Theorem 1.5 has been included because it clearly shows the essential difficulty which arises in demonstrating the results of this paper. If $x \notin \mathrm{Inv}(N, \varphi^0)$, then for all ϵ sufficiently small $x \notin \mathrm{Inv}(N, \varphi^\epsilon)$. Therefore, points of this type pose no problem and all the effort needs to be expended on understanding the behavior of the points in $\mathrm{Inv}(N, \varphi^0)$ under perturbation. In Theorem 1.5 this difficulty was defined away through the notion of slow exit and slow entrance point. Thus, what is needed are theorems which identify slow exit and slow entrance points. This is the fundamental contribution of Conley which we now describe.

We begin with some notation which will be used throughout this paper. Let $N \subset \mathbf{R}^n$ be a compact set and let $S = \mathrm{Inv}(N, \varphi^0)$.

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Let S^- (S^+) denote the set of slow exit (entrance) points. !!

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Set $S_\partial = S \cap \partial N$ and $S_\partial^{\pm} = S_\partial \cap S^{\pm}$. Let $\alpha(x, \varphi^\epsilon)$ and $\omega(x, \varphi^\epsilon)$ be the *alpha* and *omega* limit sets of x under φ^ϵ. Given an invariant set K, let $\mathcal{R}(K)$ denote the chain recurrent set of K under φ^0.

Definition 1.6 The *average of g on S*, $\mathrm{Ave}(g, S)$, is the limit as $T \to \infty$ of the set of numbers $\{\frac{1}{T}\int_0^T g(\varphi^0(s, x))\, ds \mid x \in S\}$. If $\mathrm{Ave}(g, S) \subset (0, \infty)$, then we say g has *strictly positive averages* on S.

Theorem 1.7 (Conley [2]) *$x \in S$ is a slow exit point if there exists a compact set $K_x \subset S$ invariant under φ^0, a neighborhood U_x of $\mathcal{R}(K_x)$, an $\bar{\epsilon} > 0$ and a function $\ell : \mathrm{cl}(U_x) \times [0, \bar{\epsilon}] \to \mathbf{R}$ such that the following conditions are satisfied.* !! the repeated twice

1. *$\omega(x, \varphi^0) \subset K_x$.*
2. *ℓ is of the form*

$$\ell(z, \epsilon) = \ell_0(z) + \epsilon \ell_1(z) + \ldots + \epsilon^m \ell_m(z).$$

3. *If* $L_0 = \{ z \mid \ell_0(z) = 0 \}$ *then*

$$K_x \cap \mathrm{cl}(U_x) = S \cap L_0 \cap cl(U_x)$$

and furthermore $\ell_0|_{S \cap cl(U_x)} \leq 0$.

4. *Let*

$$g_j(z) = \nabla_z \ell_0(z) \cdot f_j(z) + \nabla_z \ell_1(z) \cdot f_{j-1}(z) + \ldots + \nabla_z \ell_j(z) \cdot f_0(z).$$

Then for some m, $g_j \equiv 0$ *if* $j < m$, *and* g_m *has strictly positive averages on* $\mathcal{R}(K_x)$.

Definition 1.8 A slow exit point which satisfies the conditions of Theorem 1.7 is called a *C-slow exit point.*

Remark 1.9 It follows immediately from the definition that if x is a slow exit point for the family of flows φ^ϵ, then under the transformation $t \to -t$ (time reversal) x becomes a slow entrance point. Thus, Theorem 1.7 can, also, be used to test for slow entrance points. Slow entrance points of this form will be called *C-slow entrance points.*

<<OLD<< If moved before Definition 1.6 then no longer needed here. By the way: it seems to me that in the sequel $S^{\pm}$ is used as the notation for C-slow points

Let S^- *(*S^+*) denote the set of slow exit (entrance) points.*

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Remark 1.10 If x is a slow exit point K_x, U_x, and ℓ will denote sets and functions which satisfy the conditions of Theorem 1.7

Remark 1.11 By shrinking if necessary, we may choose U_x such that $\mathcal{R}(S) \cap U_x = \mathcal{R}(K_x)$.

Remark 1.12 At first glance Theorem 1.7 may appear to be too complicated for general applicability. However, it should be observed that the only dynamics which plays a role in the calculations is that of φ^0. Since this is the singular flow one expects it to be simple and, in fact, for the series of examples which will be presented in the last section of this paper, K_x consists of isolated fixed points. Thus $\mathcal{R}(K_x)$ is also a fixed point, and hence $\mathrm{Ave}(g, K_x)$ is easy to compute.

With the knowledge of how to identify singular isolating neighborhoods we now turn to the question of how to characterize singular index pairs. To adequately motivate the construction given in Theorem 1.16 we need to recall the definition of an index pair.

Definition 1.13 Let S be an isolated invariant set. A pair of compact sets (N, L) where $L \subset N$ is called an *index pair* for S if:

1. $S = \mathrm{Inv}(\mathrm{cl}(N \setminus L))$ and $N \setminus L$ is a neighborhood of S;
2. L is *positively invariant* in N, i.e. given $x \in L$ and $\varphi([0,t],x) \subset N$, then $\varphi([0,t],x) \subset L$
3. L is an *exit set* for N, i.e. given $x \in N$ and $t_1 > 0$ such that $\varphi(t_1, x) \notin N$, then there exists $t_0 \in [0, t_1]$ for which $\varphi([0,t_0],x) \subset N$ and $\varphi(t_0, x) \in L$.

With this in mind the singular index pair (N, L) will be constructed in such a manner as to mimic the above three conditions as closely as possible. With regard to the exit set requirement it is clear that L will need to contain the *immediate exit* set for N which is defined by

$$N^- := \{x \in \partial N \mid \varphi^0((0,t),x) \not\subset N \text{ for all } t > 0\}.$$

Positive invariance leads to the following construction. Given $Y \subset N$ its *push forward set* in N under the flow φ^0 is defined to be

$$\rho(Y, N, \varphi^0) := \{x \in N \mid \exists z \in Y,\, t \geq 0 \text{ such that } \varphi^0([0,t],z) \subset N,\, \varphi^0(t,z) = x\}.$$

Observe that $\rho(Y, N, \varphi^0)$ consists of the set of points which can be reached from Y by a positive trajectory in N and that $Y \subset \rho(Y, N, \varphi^0)$. The *unstable set* of an invariant set $Y \subset N$ under φ^0 is

$$W^u_N(Y) := \{x \in N \mid \varphi^0((-\infty,0),x) \subset N \text{ and } \alpha(x,\varphi^0) \subset Y\}.$$

Notice that $Y \subset W_N^u(Y)$. Since we will only be interested in the unstable set under the singular flow φ^0 we do not include it in the notation.

Remark 1.14 It is easy to find examples of isolating neighborhoods N for which there does not exist a set L such that (N, L) is an index pair for Inv$N$. To prevent us from considering such N we will use the following condition.

Definition 1.15 A slow entrance point x is a *strict slow entrance point* if there exists a neighborhood Θ_x of x and an $\bar{\epsilon} > 0$ such that if $y \in \Theta_x \cap N$ and $\epsilon \in (0, \bar{\epsilon}]$, then there exists $t_y(\epsilon) > 0$ for which

$$\varphi^\epsilon([0, t_y(\epsilon)], y) \subset N.$$

We will let S_∂^{++} denote the strict slow entrance points.

Theorem 1.16 *Let N be a singular isolating neighborhood. Assume*

1. $S_\partial \subset S_\partial^{++} \cup S_\partial^-$.
2. $(S_\partial^{++} \setminus S_\partial^-) \cap \mathrm{cl}(N^-) = \emptyset$.

For each $x \in S_\partial^-$, let K_x denote a compact invariant set as in Theorem 1.7. Define

$$L := \rho(cl(N^-), N, \varphi^0) \bigcup W_N^u \left(\bigcup_{x \in S_\partial^-} \mathcal{R}(K_x) \right).$$

If L is closed, then (N, L) is a singular index pair for the family of flows φ^ϵ.

The proof of this theorem will appear in the next section. In Section 3 two examples will be presented which remark on the necessity of assumptions 1.16.1 and 1.16.2. Finally we conclude with some simple applications in Section 4. These applications are included *only* to indicate how Theorem 1.16 can be applied. The results themselves are classical and have been studied in much greater detail by other authors too numerous to include here. More serious applications of this theorem will be presented in future papers.

2 Proof of Theorem 1.16

The difficulty in the proof of Theorem 1.16 is that in general, it is almost certain that (N, L) is not an index pair for φ^ϵ no matter how small $\epsilon > 0$ is. Thus, for small ϵ it is necessary to construct an index pair of the form (N, L^ϵ) and then relate $H^*(N, L)$ to $H^*(N, L^\epsilon)$. However, constructing index pairs is notoriously difficult. Thus, we need to control how the flow changes as a function of ϵ. Our idea is to restrict the perturbation to an η-neighborhood of $\cup_{x \in S_\partial} \mathcal{R}(K_x)$ using bump functions. For η small, we are able to construct a singular index pair. Furthermore, the resulting family of index pairs limits to the singular pair (N, L) as $\eta \to 0$. The weak continuity of the Alexander–Spanier cohomology can now be applied to prove directly that (N, L) is a singular index pair for small η. Finally, with ϵ fixed η is varied until η is larger than the diameter of N and we have the original perturbation problem. Since N is a singular isolating neighborhood for all η, the continuation property of the index implies that the Conley index is independent of η, and hence, (N, L) is a singular index pair for the original family of flows φ^0.

We begin with some preliminary results concerning slow exit points. Throughout this section we will assume that the hypotheses of Theorem 1.16 are satisfied.

Lemma 2.1 *Let $x \in S$ be a C-slow exit point and let $y \in \mathcal{R}(K_x)$. Then y is a C-slow exit point.*

Proof: Since $y \in \mathcal{R}(K_x) \subset S$, $y \in S$. Furthermore, $\omega(y, \varphi^0) \subset K_x$. Thus we can choose $K_y = K_x$, $U_y = U_x$ and the same function ℓ. ∎

The next result is essentially a corollary of the proof of [2, Lemma 3.2.B].

Proposition 2.2 *Let x be a C-slow exit point. Then, there exists a neighborhood U_x of $\mathcal{R}(K_x)$ as in the definition of a C-slow exit point and there exists a neighborhood W of S with the following property. Given $y \in \mathcal{R}(K_x)$ and $\epsilon > 0$ sufficiently small, there exist numbers (depending on ϵ), $0 < t_0(\epsilon) \le t_1(\epsilon)$ such that*

$$\begin{aligned} \varphi^\epsilon([0, t_0(\epsilon)], y) &\subset U_x \\ \varphi^\epsilon([t_0(\epsilon), t_1(\epsilon)], y) \cap W &= \emptyset \\ \varphi^\epsilon(t_1(\epsilon), y) &\notin N. \end{aligned}$$

Notice that by shrinking U_x if necessary, we may assume, as in Remark 1.11, that $\mathcal{R}(S) \cap U_x = \mathcal{R}(K_x)$.

Corollary 2.3 *Let x be a C-slow exit point. Let $\{y_i\} \subset N$ such that $y_i \to y \in \mathcal{R}(K_x)$. Given a sequence of parameter values ϵ_i converging to 0, let*

$$\Gamma_i = \varphi^{\epsilon_i}([0, t_i], y_i) \subset N$$

and assume that $\Gamma_i \to \Gamma$ in the Hausdorff topology. Then

$$\Gamma \subset W^u(\mathcal{R}(K_x)).$$

Proof:

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Let U_x be an open set of $\mathcal{R}(K_x)$ as determined by Proposition 2.2, and satisfying Remark 1.11. Let $z_i = \varphi^{\epsilon_i}(t'_i, y_i)$ for some $t'_i \in [0, t_i]$. Obviously $z_i \in \Gamma_i$. Assume $z_i \to z$. If $z \in U_x$, then clearly $t'_i < t_1(\epsilon_i)$ (where $t_1(\epsilon_i)$ comes from Proposition 2.2). If the $t'_i < t_0(\epsilon_i)$, then $\varphi^{\epsilon_i}([0, t'_i], y_i) \subset U_x$ for all i. In this case, either $z \in \mathcal{R}(K_x)$, or $\alpha(z, \varphi^0) \subset U_x$. Since $\alpha(z, \varphi^0)$ is a chain recurrent subset of S, Remark 1.11 implies $\alpha(z, \varphi^0) \subset \mathcal{R}(K_x)$. If we have $t'_i > t_0(\epsilon_i)$, then by Proposition 2.2 there is a neighborhood W of S such that $\varphi^{\epsilon_i}([t_0(\epsilon_i), t'_i], y_i) \cap W = \emptyset$. It follows that there is a $T > 0$ such that $t'_i - t_0(\epsilon_i) < T$. We can apply the argument of the preceding paragraph to $z'_i = \varphi^{\epsilon_i}(-T, z_i)$ and again conclude that $\alpha(z, \varphi^0) \subset \mathcal{R}(K_x)$.

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Choose $U_x, W, t_0(\epsilon_i), t_1(\epsilon_i)$ as in Propoistion 2.2. Additionally assume U_x is small enough to satisfy the conclusion of Remark 1.11. Let $z \in \Gamma$. Then $z = \lim z_i$, where $z_i = \varphi^{\epsilon_i}(t'_i, y_i)$ for some $t'_i \in [0, t_i]$. If $z \in U_x$, then clearly $t'_i < t_1(\epsilon_i)$. If the $t'_i < t_0(\epsilon_i)$, then $\varphi^{\epsilon_i}([0, t'_i], y_i) \subset U_x$ for all i. In this case, either $z \in \mathcal{R}(K_x)$ if the numbers $\{t'_i\}$ are bounded away from infinity, or $\alpha(z, \varphi^0) \subset U_x$ otherwise. Since $\alpha(z, \varphi^0)$ is a chain recurrent subset of S, Remark 1.11 implies $\alpha(z, \varphi^0) \subset \mathcal{R}(K_x)$. If we have $t'_i > t_0(\epsilon_i)$, then $\varphi^{\epsilon_i}([t_0(\epsilon_i), t'_i], y_i) \cap W = \emptyset$. It follows that there is a $T > 0$ such that $t'_i - t_0(\epsilon_i) < T$. We can apply the argument of the preceding paragraph to $z'_i = \varphi^{\epsilon_i}(-T, z_i)$ and again conclude that $\alpha(z, \varphi^0) \subset \mathcal{R}(K_x)$. !!

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We shall also make use of the following result which says that limit points of N^- either leave N eventually in forward time, or pass near $\cup_{x\in S_\partial^-}\mathcal{R}(K_x)$ under φ^0.

Lemma 2.4 *Let V be a neighborhood of $\cup_{x\in S_\partial^-}\mathcal{R}(K_x)$. Let $x \in cl(N^-)$. Then, either*

1. *$\varphi^0([0,\infty),x) \not\subset N$, or*
2. *there exists $t_x > 0$ such that $\varphi^0([0,t_x],x) \subset N$ and $\varphi^0(t_x,x) \in V$.*

Proof: The proof is by contradiction. So assume $\varphi^0([0,\infty),x) \subset N$ and $\varphi^0([0,\infty),x) \cap V = \emptyset$. Let $y \in \omega(x,\varphi^0)$. Clearly $y \in \mathrm{cl}(\rho(\mathrm{cl}(N^-),N,\varphi^0)) \subset \mathrm{cl}L$. Since L is assumed to be closed, $y \in L$.

Assume that $y \in W^u(\cup_{x\in S_\partial^-}\mathcal{R}(K_x))$. Then $\varphi^0((-\infty,T),y) \subset V$ for some T. Let V' be a neighborhood of $\varphi^0(T,y)$ such that $V' \subset V$. Since $y \in \omega(x,\varphi^0)$, for sufficiently large T_x, $\varphi(T_x,x)$ can be assumed to be arbitrarily close to y. In particular, T_x can be chosen large enough that $t_x = T_x - T$ implies $\varphi^0(t_x,x) \in V' \subset V$.

The remaining case that needs to be considered is $y \in \rho(\mathrm{cl}(N^-),N,\varphi^0)$, but $y \notin W^u(\cup_{x\in S_\partial^-}R(K_x))$. Since $y \in \rho(\mathrm{cl}(N^-),N,\varphi^0)$, there exists $z \in \mathrm{cl}(N^-)$ such that for some $t_0 > 0$, $\varphi^0(t_0,z) = y$. Since $y \in \omega(x,\varphi^0) \subset S$,

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$z \in S \cap \partial N \cap \mathrm{cl}(N^-)$.

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$z \in S \cap \mathrm{cl}(N^-)$. !!

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By assumption 2 of Theorem 1.16, $z \in S_\partial^-$ and hence there is a $T > 0$ such that $\varphi^0(T,z) \in V$. Since $z \in \omega(x,\varphi^0)$, there exists $t_x > 0$ such that $\varphi^0([0,t_x],x) \subset N$ and $\varphi^0(t_x,x) \in V$. ∎

Lemma 2.5 $\cup_{x\in S_\partial^-}\mathcal{R}(K_x)$ *is compact.*

Proof: First notice that if $z \in S_\partial^-$ and z is chain recurrent, then $z \in \cup_{x\in S_\partial^-}\mathcal{R}(K_x)$. This follows because $\omega(z,\varphi^0) \subset \mathcal{R}(K_z)$, and for any ϵ, T, there is clearly an ϵ, T chain from z to any point in $\omega(z,\varphi^0)$.

Now consider a sequence $\{z_n\} \subset \cup_{x\in S_\partial^-}\mathcal{R}(K_x)$ with $z_n \to z$. Note that $z \in \mathcal{R}(S)$ since a chain recurrent set is closed, and by the assumption on L, $z \in L$. We must show that $z \in \cup_{x\in S_\partial^-}\mathcal{R}(K_x)$.

Since z is chain recurrent, if $z \in W^u(\cup_{x\in S_\partial^-}\mathcal{R}(K_x))$ then $z \in \cup_{x\in S_\partial^-}\mathcal{R}(K_x)$ Since $z \in L$, we only need to consider $z \in \rho(\mathrm{cl}(N^-), N, \varphi^0)$. So there is $y \in S_\partial$ and $t \geq 0$ with $z = \varphi^0(t,y)$. By assumption 2 of Theorem 1.16, $y \notin S_\partial^{++}$, !! THeorem- ¿Theorem so $y \in S_\partial^-$, and hence $z \in S_\partial^-$. By the observation at the beginning of the proof, $z \in \cup_{x\in S_\partial^-}\mathcal{R}(K_x)$. ∎

We now begin constructing the bump functions. Given $x \in \mathbf{R}^n$ and $\eta > 0$, let $B_\eta(x) := \{y \in \mathbf{R}^n \mid ||x-y|| < \eta\}$ and given $Y \subset \mathbf{R}^n$, let $B_\eta(Y) := \cup_{y\in Y} B_\eta(y)$. Define $Q_\eta^- = B_\eta(\cup_{x\in S_\partial^-}\mathcal{R}(K_x))$, $Q_\eta^+ = B_\eta(S_\partial^{++})$, and $Q_\eta = Q_\eta^- \cup Q_\eta^+$. Our family of bump functions will consist of smooth functions $\mu_\eta : \mathbf{R}^n \to [0,1]$ satisfying

- supp $\mu_\eta \subset Q_\eta$.
- $B_{\eta/2}(\cup_{x\in S_\partial^-}\mathcal{R}(K_x)) \subset \mu^{-1}(1)$.

We now consider the two parameter singular perturbation problem given by the equation

$$\dot{x} = f(x,\epsilon) = f_0(x) + \mu_\eta(x)[\sum_{i=1}^{m} \epsilon^i f_i(x) + o(\epsilon^m)] \qquad (2)$$

and let ψ_η^ϵ denote its flow. Notice that for η sufficiently large, $\psi_\eta^\epsilon = \varphi^\epsilon$.

Lemma 2.6 *Assume that S_∂ consists of C-slow entrance and exit points, and let r be the diameter of N. Then there is a continuous function $\tilde{\epsilon}$: $(0,r] \to (0,\infty)$ with the propety that N is an isolating neighborhood under* !! under *ψ_η^ϵ for all (ϵ,η) such that $0 < \eta \leq r$ and $0 < \epsilon \leq \tilde{\epsilon}(\eta)$.*

Proof: For convenience we assume $r > 1$, and let n be a positive integer. From a parameterized version of Theorem 1.7, (see [10]) it follows that for $\eta \in$

$[\frac{1}{n}, r]$, there are $\epsilon_\eta > 0$ and $\mu_\eta > 0$ such that for $(\lambda, \epsilon) \in (\eta - \mu_\eta, \eta + \mu_\eta) \times (0, \epsilon_\eta]$, N is an isolating neighborhood under ψ_λ^ϵ. Clearly we may choose $\epsilon_{n+1} < \epsilon_n$, and $\lim_{n\to\infty} \epsilon_n = 0$. Let $\tilde{\epsilon}$ be a piecewise linear function with $\tilde{\epsilon}(\frac{1}{n}) = \epsilon_{n+1}$. ∎ !! under

We now construct an index pair for ψ_η^ϵ for small η and ϵ. Let

$$L_\eta^\epsilon = \rho(\mathrm{cl}(Q_\eta^-), N, \psi_\eta^\epsilon) \cup \rho(\mathrm{cl}(N^-), N, \varphi^0). \tag{3}$$

Lemma 2.7 *There exists $\bar{\eta} > 0$ such that given $\eta \in (0, \bar{\eta}]$, there is an $\bar{\epsilon}(\eta) > 0$ such that for $\epsilon \in (0, \bar{\epsilon}(\eta)]$,*

$$(N, L_\eta^\epsilon)$$

is an index pair for ψ_η^ϵ.

Proof: The proof consists of checking that the three conditions of Definition 1.13 are satisfied.

(Step 1) We shall show that L_η^ϵ is closed (and hence compact) by showing that every point of L_η^ϵ leaves in a finite positive time. We begin by showing that $\rho(\mathrm{cl}(Q_\eta^-), N, \psi_\eta^\epsilon)$ is closed. By Lemma 2.1, if $y \in \cup_{z \in S_\partial^-} \mathcal{R}(K_z)$, then y is a C-slow exit point. Hence, there exists U_y a neighborhood of y and an $\bar{\epsilon}_y(\eta)$ such that for every $\epsilon \in (0, \bar{\epsilon}_y(\eta))$ there exists $T_y(\epsilon, \eta) > 0$ such that !! $\bar{\epsilon}_y(\eta)$

!! $T_y(\epsilon, \eta) > 0$

$$\psi_\eta^\epsilon(T_y(\epsilon, \eta), U_y) \cap N = \emptyset.$$

!! $T_y(\epsilon, \eta) > 0$

Since by Lemma 2.5 $\cup_{z \in S_\partial^-} \mathcal{R}(K_z)$ is compact, it can be covered by a finite collection $\{U_{y_1}, \ldots, U_{y_k}\}$. Choose $\bar{\eta} > 0$ such that

<<BEG<< Here is my problem: U_{y_i} depend on η (unless we are able to prove otherwise). Thus, since both sides depend on η the inclusion may never be satisfied.

$\mathrm{cl}Q_\eta^- \subset \cup_{i=1}^k U_{y_i}$. !!

>>END>>

Let $\bar{\epsilon} := \min_{1 \le i \le k} \bar{\epsilon}_{y_i}$ and $T(\epsilon) := \max_{1 \le i \le k} T_{y_i}(\epsilon)$. Then for all $w \in \mathrm{cl}Q_\eta^-$, !! $\mathrm{cl}Q_\eta^-$

$$\psi_\eta^\epsilon([0, T(\epsilon)], w) \not\subset N,$$

i.e., each point in $\mathrm{cl}Q_\eta^-$ leaves in a finite forward time. This implies that $\rho(\mathrm{cl}(Q_\eta^-), N, \psi_\eta^\epsilon)$ is closed. !! W-¿w !! $\mathrm{cl}Q_\eta^-$

Now consider the set $\rho(\mathrm{cl}(N^-), N, \varphi^0)$. Let $x \in \rho(\mathrm{cl}(N^-), N, \varphi^0)$. Then $x = \varphi^0(t_{xy}, y)$ for some $t_{xy} > 0$ and some $y \in \mathrm{cl}(N^-)$, By Lemma 2.4 there exists $T_y > 0$ such that $\varphi^0(T_y, y) \notin N$ or $\varphi^0(T_y, y) \in Q_\eta^-$. In either case, this implies the existence of a neighborhood V_y of y such that $\varphi^0(T_y, V_y) \cap N = \emptyset$ or $\varphi^0(T_y, V_y) \subset Q_\eta^-$, respectively. Thus, x either enters Q_η^- or leaves N in finite time. Therefore, L_η^ϵ is closed and condition 1 of Definition 1.13 is satisfied.

(Step 2) L_η^ϵ is positively invariant. For points in $\rho(\mathrm{cl}(Q_\eta^-), N, \psi_\eta^\epsilon)$ this is clear. For points in $\rho(\mathrm{cl}(N^-), N, \varphi^0)$, this follows from the arguments in Step 1. If such points stay out of Q_η^- in forward time, then $\psi_\epsilon^\eta = \phi^0$, and if they enter Q_η^- then they stay in L_η^ϵ.

(Step 3) We now show that L_η^ϵ is an exit set. Let $x \in N$ and assume for some $(\epsilon, \eta) \in (0, \bar{\epsilon}] \times (0, \bar{\eta}]$ that $\psi_\eta^\epsilon(t_0, x) \notin N$. If

$$\psi_\eta^\epsilon([0, t_0], x) \cap Q_\eta = \emptyset,$$

then there exists $t_1 \in [0, t_0]$ such that $\psi_\eta^\epsilon(t_1, x) \in N^-$ and $\psi_\eta^\epsilon([0, t_1], x) \subset N$ since

$$\psi_\eta^\epsilon|_{N \setminus Q_\eta} = \varphi^\epsilon|_{N \setminus Q_\eta}.$$

So assume the forward trajectory does not leave through $N \setminus Q_\eta$. For $\bar{\eta} > 0$ sufficiently small no orbit can exit through Q_η^+ since the slow exit points are, in fact, strict slow exit points. Finally, if x leaves through Q_η^-, then x leaves through L_η^ϵ. ∎

From now on we assume $\bar{\epsilon}(\eta)$ is such as to satisfy Lemma 2.7 and to assure that

$$W^u(\cup_{x \in S_\partial^-} \mathcal{R}(K_x)) \subset \rho(Q_\eta^-, N, \psi_\eta^\epsilon) \quad \text{for} \quad \epsilon \leq \bar{\epsilon}(\eta). \tag{4}$$

This latter condition can be satisfied since L is compact and positively invariant. !! $\epsilon -> \bar{\epsilon}$

The next lemma allows us to choose a decreasing sequence of parameter values η_i such that $L_{\eta_i}^{\epsilon(\eta_i)}$ is a nested sequence.

Lemma 2.8 *There is a sequence η_i decreasing to 0 and a choice $\epsilon(\eta_i) \in (0, \bar{\epsilon}(\eta_i)]$ such that $L_{\eta_{i+1}}^{\epsilon(\eta_i+1)} \subset L_{\eta_i}^{\epsilon(\eta_i)}$.*

Proof: We will show that given η, for $\nu > 0$ small enough,

$$\rho(Q_{\nu\eta}^-, N, \psi_{\nu\eta}^{\epsilon(\nu\eta)}) \subset \rho(Q_\eta^-, N, \psi_\eta^\epsilon).$$

Suppose this is not the case. Then for every $\nu > 0$ there are $t_\nu > 0$ and $y_\nu \in Q^-_{\nu\eta}$ for which $\psi^{\epsilon(\eta\nu)}_{\nu\eta}([0,t_\nu], y_\nu) \subset N$ and $\psi^{\epsilon(\eta\nu)}_{\nu\eta}(t_\nu, y_\nu) = z_\nu \notin \rho(Q^-_\eta, N, \psi^\epsilon_\eta)$. Obviously, $\psi^{\epsilon(\eta\nu)}_{\nu\eta}(-t_\nu, z_\nu) = y_\nu$. Let $\nu \to 0$. Let y be a limit point of the y_ν and z a limit point of z_ν as $\nu \to 0$. Since $y \in \cap_{\nu>0} Q^-_\eta$, $y \in \cup_{x\in S^-_\partial} \mathcal{R}(K_x)$. By Corollary 2.3, $z \in W^u(\cup_{x\in S^-_\partial}\mathcal{R}(K_x))$. But by (4) this implies that $z \in \rho(Q^-_\eta, N, \psi^\epsilon_\eta)$, a contradiction.

Now start with η_1, and let $\eta_2 = \nu\eta_1$ where ν is such that $\rho(Q^-_{\nu\eta_1}, N, \psi^{\epsilon(\nu\eta_1)}_{\nu\eta_1}) \subset \rho(Q^-_\eta, N, \psi^\epsilon_\eta)$. Inductively, let $\eta_{i+1} = \nu\eta_i$ with ν chosen appropriately. ∎

Notice that the above proof shows we may choose $\epsilon(\eta_i) < \tilde{\epsilon}(\eta_i)$, where $\tilde{\epsilon}$ is the function from Lemma 2.6.

Finally, we show that the $L^{\epsilon(\eta_i)}_{\eta_i}$ "converge" to L.

Lemma 2.9 $\cap_{i\geq 1} L^{\epsilon(\eta_i)}_{\eta_i} = L$.

Proof: The compactness of L,i.e. (4) implies that $L \subset L^{\epsilon(\eta_i)}_{\eta_i}$ for i sufficiently large, so $L \subset \cap_{i\geq 1} L^{\epsilon(\eta_i)}_{\eta_i}$.

On the other hand, suppose $z \in \cap_{i\geq 1} L^{\epsilon(\eta_i)}_{\eta_i}$. Then for each i, $z = \psi^{\epsilon(\eta_i)}_{\eta_i}(t_i, y_i)$ for some $y_i \in Q^-_\eta$, $t_i \geq 0$. By taking a subsequence if necessary, we may assume that $y_i \to y \in \cup_{x\in S^-_\partial}\mathcal{R}(K_x)$ and the result follows from Corollary 2.3 ∎

We can now prove Theorem 1.16.

Proof: Let $G = \{\,(\eta,\epsilon) \mid 0 < \eta \leq r,\ 0 < \epsilon \leq \tilde{\epsilon}(\eta)\,\}$. By the weak continuity property of the Alexander–Spanier cohomology ([7], [13]), the inclusion maps $\iota_i : (N, L) \to (N, L^{\epsilon(\eta_i)}_{\eta_i})$ induce an isomorphism

$$\lim_{\rightarrow} H^*(N, L^{\epsilon(\eta_i)}_{\eta_i}) \approx H^*(N, L).$$

On the other hand, by the standard continuation theorem for the Conley index, for $(\eta,\epsilon), (\eta',\epsilon') \in G$ we have

$$CH^*(\mathrm{Inv}(N, \psi^\epsilon_\eta)) \approx CH^*(\mathrm{Inv}(N, \psi^{\epsilon'}_{\eta'})),$$

i.e.

$$H^*(N, L^\epsilon_\eta) \approx H^*(N, L^{\epsilon'}_{\eta'}),$$

so

$$H^*(N, L^{\tilde{\epsilon}(r)}_r) \approx H^*(N, L)$$

and (N, L) is a singular index pair. ∎

Figure 1: A singular isolating neighborhood which cannot be made into an index pair.

3 Examples

We now present two examples which are meant to indicate why conditions 1 and 2 are included in Theorem 1.16.

Example 3.1 In Figure 1 the shaded region is the singular isolating neighborhood N. The solid flow lines indicate the singular flow φ^0 while the dashed lines are the flow lines of φ^ϵ for $\epsilon > 0$. The degenerate fixed point $\bar{x}$ is a slow entrance point but not a strict slow entrance point. Using the definition of equation (3) one concludes that

$$L_\eta^\epsilon = \emptyset.$$

Of course this is not an exit set for N under ψ_η^ϵ. Thus the proof of Theorem 1.16 will not work. However, the index computation $H^*(N, L_\eta^\epsilon)$ is correct. This raises the question of whether condition 1.16.1 is necessary.

Example 3.2 In Figure 2 the singular isolating neighborhood N consists of the cylindrical tube and the two flat regions indicated. The boundary

Figure 2: $S_\partial^{++} \cap \mathrm{cl}(N^-) = \emptyset$ is necessary.

of the cylinder is invariant and limits onto the degenerate periodic orbit. The shaded regions and dark lines indicate L as defined by Theorem 1.16. Under perturbation, the degenerate periodic orbit disappears and $\mathrm{Inv}(N, \varphi^\epsilon)$ consists of an attracting fixed point. Thus $H^*(N, L) \not\approx CH^*(\mathrm{Inv}(N, \varphi^\epsilon))$.

Figure 3: The phase portraits of equation (4.1) at different parameter values.

4 Applications

As was indicated in the introduction, the applications presented in this section are included only to demonstrate how Theorems 1.7 and 1.16 can be used. More compelling applications will be presented in future work.

4.1 A heteroclinic orbit

We start with a continuous parameterized family of flows generated by

$$\dot{x} = g(x, \lambda), \quad x \in \mathbf{R}^2, \quad \lambda \in \Lambda \tag{5}$$

where Λ is an open interval containing $[-1, 1]$. For each parameter value (5) has two saddle points p and q whose stable and unstable manifolds are as indicated in Figure 3. Furthermore, it is assumed that there exists a connecting orbit from p to q only at $\lambda = 0$.

We now consider the family of flows

$$\varphi^\epsilon : \mathbf{R} \times \mathbf{R}^2 \times \Lambda \to \mathbf{R}^2 \times \Lambda$$

generated by

$$\begin{aligned} \dot{x} &= g(x,\lambda) \\ \dot{\lambda} &= \epsilon(\lambda+1)(\lambda-1) \end{aligned} \tag{6}$$

Observe that in the above system $f_0(x,\lambda) = (g(x,\lambda),0)$ and $f_1(x,\lambda) = (0,(\lambda+1)(\lambda-1))$.

Proposition 4.1 *For $\epsilon > 0$, sufficiently small, equation (6) admits a heteroclinic orbit from $(p,1)$ to $(q,-1)$.*

The proof of this proposition will be done in four steps. The first, is to identify the desired compact set N. The second, is to use the conditions of Theorem 1.7 to verify that N is a singular isolating neighborhood. The third step is to use Theorem 1.16 to compute the Conley index of $\mathrm{Inv}(N,\varphi^\epsilon)$. Finally, we use this index to conclude the existence of the heteroclinic orbit.

(Step 1) From our knowledge of the one parameter family of flows given by equation (5), it is reasonable to guess that if there exists a connecting orbit from $(p,1)$ to $(q,-1)$ then it lies close to the set of points

$$\{p\}\times[0,1] \bigcup \{q\}\times[-1,0] \bigcup \text{connecting orbit } p\to q. \tag{7}$$

With this in mind we choose two sets $N_p \subset \mathbf{R}^2$ and $N_q \subset \mathbf{R}^2$ containing the points p and q, respectively, as indicated in Figure 4. Let $\lambda_1 < 0 < \lambda_2$. For small positive δ, let

$$N = N_p \times [\lambda_1, 1+\delta] \cup N_q \times [-1-\delta, \lambda_2].$$

It is easy to see that the set defined in (7) is a subset of N.

(Step 2) Let $S = Inv(N,\varphi^0)$. Then

$$S_\partial = \{(p,\lambda_1),(p,1+\delta),(q,-1-\delta),(q,\lambda_2)\}.$$

To see that (p,λ_1) is a slow exit point choose $K_{(p,\lambda_1)} = (p,\lambda_1)$, $U_{(p,\lambda_1)}$ to be any small neighborhood of (p,λ_1), and $\ell(x,l) = l$. Similar arguments show that $(q,-1-\delta)$ is also a slow exit point, while $(p,1+\delta)$ and (q,λ_2) are strict slow entrance points.

Figure 4: The sets N_p and N_q with flows at parameter values $\lambda_1 < 0$ and $\lambda_2 > 0$. The dashed line represents the forward orbit of the lower left corner of N_p under the flow φ^{l_2}. $\mathcal{R}$ denotes the "triangular" region in the upper right corner of N_q bounded by the dashed orbit.

(Step 3) Using the singular flow φ^0 it is easy to check that

$$\begin{aligned} N^- &= N_p^{top} \times [\lambda_1, 1+\delta] \bigcup N_p^{bottom} \times (\lambda_2, 1+\delta] \bigcup \\ & N_q^{left} \times [-1-\delta, \lambda_2] \bigcup N_q^{right} \times [-1-\delta, \lambda_2]. \end{aligned}$$

Taking the closure of this adds $N_p^{bottom} \times \{\lambda_2\}$. Now we take the forward orbits of this in N which adds $\mathcal{R} \times \{\lambda_2\}$. To complete the description of L we need to add the slow exit points and their unstable sets. The unstable set for (p, λ_1) (in the flow parameterized by λ_1 is a curve Γ_1 connecting $N_p^{top} \times \{\lambda_1\}$ to $N_q^{left} \times \{\lambda_1\}$. Similarly, the unstable set for $(p, 1+\delta)$ is a curve Γ_2 connecting $N_p^{top} \times \{1+\delta\}$ to $N_p^{bottom} \times \{1+\delta\}$. Putting this all together gives

$$\begin{aligned} L &= N_p^{top} \times [\lambda_1, 1+\delta] \cup N_p^{bottom} \times (\lambda_2, 1+\delta] \cup \mathcal{R} \times \{\lambda_2\} \cup \\ & N_q^{left} \times [-1-\delta, \lambda_2] \cup N_q^{right} \times [-1-\delta, \lambda_2] \cup \Gamma_1 \cup \Gamma_2. \end{aligned}$$

L is contractible in N and N is, also, contractible so

$$H^*(N, L) \approx 0.$$

By Theorem 1.16 we can conclude that $CH(\mathrm{Inv} N, \varphi^\epsilon) = 0$ for small positive ϵ.

(Step 4) Observe that λ is a Lyaponov function on N for φ^ϵ if $\epsilon > 0$. Thus, $((q, -1-\delta), (p, 1+\delta))$ is an attractor–repeller decomposition of $\mathrm{Inv}(N, \varphi^\epsilon)$. Since $(q, -1-\delta)$ has a one–dimensional unstable manifold and $(p, 1+\delta)$ has a two dimensional unstable manifold,

$$CH^k\left((p, 1+\delta) \cup (q, -1-\delta)\right) \approx \begin{cases} \mathbf{Z} & \text{if } k = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

But this implies that

$$CH^*\left((p, 1+\delta) \cup (q, -1-\delta)\right) \not\approx CH(\mathrm{Inv} N, \varphi^\epsilon)$$

for $\epsilon > 0$ and therefore the desired connecting orbit exits.

4.2 A periodic orbit in $\mathbf{R}^2$

(See [2]). Consider the fast-slow system in the plane

$$\begin{aligned} \dot{x} &= \epsilon g(x, y) \\ \dot{y} &= h(x, y). \end{aligned} \tag{8}$$

Figure 5: A singular isolating neighborhood for the planar system.

The sets $\{g = 0\}$ and $\{h = 0\}$, which intersect at the origin, are shown in solid lines in Figure 5. For simplicity, we assume that $\frac{\partial g}{\partial x} < 0$, $\frac{\partial g}{\partial y} < 0$ and $\frac{\partial h}{\partial x} > 0$

Our first goal is to show that for $\epsilon \in (0, \bar{\epsilon}]$ the set of bounded solutions of φ^ϵ consists of more than just the equilibrium point at the origin. To do this let N be the dashed rectangle in Figure 5.

Observe that $S_0 \cap \partial N$ consists of two points, labelled $p = (p_1, p_2)$ and $q = (q_1, q_2)$, where the $\{h = 0\}$ curve intersects ∂N. To see that p is a slow exit point, we argue as in step 2 of the previous example, namely, we take $K_p = p$, U_p to be any small neighborhood of p, and $\ell(x, y) = x$. The sign of g implies that p is a slow exit point, and the sign of h implies that its unstable set in the singular flow is the left boundary segment of N. Similarly, q is a slow exit point, and the sign of h implies that its unstable set in the singular flow is the right boundary segment of N. Finally, the sign of h implies that the top and bottom of N compose N^-, so $L = \partial N$. Thus, by Theorem 1.16 $(N, \partial N)$ is a singular index pair, and hence, for $\epsilon \in (0, \bar{\epsilon}]$

$$CH^k(\mathrm{Inv}(N, \varphi^\epsilon)) \approx H^k(N, \partial N) \approx \begin{cases} \mathbf{Z} & \text{if } k = 2; \\ 0 & \text{otherwise.} \end{cases}$$

However, the origin is an attracting fixed point for $\epsilon > 0$. Clearly, the Conley index of $\mathrm{Inv}(N, \varphi^\epsilon)$ does not equal the Conley index of the fixed point at the origin. Therefore, $\mathrm{Inv}(N, \varphi^\epsilon)$ contains more than just the fixed point at the origin.

The simplicity of this proof suggests that arguments of this form, which provide for the existence of non–trivial invariant sets, can be carried out in higher dimensional problems without too much difficulty. Of course, for this two dimensional problem one can appeal to the Poincaré–Bendixson theorem to conclude that $\mathrm{Inv}(N, \varphi^\epsilon)$ contains a periodic orbit. There are at least two reasons why we don't want to do this:

1. we do not obtain good bounds on the location of the periodic orbit,

2. the argument cannot, in general, be carried over to higher dimensional problems.

With this in mind, we now provide another proof of the existence of a periodic orbit for $\epsilon \in (0, \bar{\epsilon}]$. This proof is based on a more judicious choice of N and a more sophisticated use of the Conley index.

Let N be the shaded region in Figure 6. It is left to the reader to check that N is a singular isolating neighborhood. Thus, we begin by determining L. The same argument as above shows that L contains the entire outer boundary of N. We will show that the inner boundary of N is also in L. Notice that the inner boundary of N consists of three different types of points:

1. The horizontal segments on the top and bottom.

2. The curved portions in $S_0 \cap \partial N$ on the top and bottom. These points are critical points in the singular flow.

3. The vertical segments in $S_0 \cap \partial N$ on the left and right.

The sign of h implies that the points in the first group are in N^-. The points in the second group are slow exit points. As usual, for any such point z, use $K_z = z$, U_z a small neighborhood, and $\ell(x, y) = x$. The points in the third group, those on the vertical segments, are also slow exit points. To see this, let v be the point at the bottom on the left vertical segment, i.e. v is a critical point for the singular flow, hence also in group 2 in the above list. So we have $K_v = v$, and U_v a small neighborhood of v, and $\ell(x, y) = x$. For any

Figure 6: A refined singular isolating neighborhood N.

z on the left vertical segment, use $K_z = K_v = v$, $U_z = U_v$, and $\ell(x, y) = x$. We have $\omega(z) = v$ in the singular flow, so by Theorem 1.7, z is a slow exit point. A similar argument shows that the left vertical segment consists of slow exit points. Thus, all points on the inner boundary of N are either in slow exit points or points in N^-, so the inner boundary is also contained in L. It follows that $L = \partial N$, thus for $\epsilon \in (0, \bar{\epsilon}]$,

$$CH(\mathrm{Inv}(N, \varphi^\epsilon)) \approx \begin{cases} \mathbf{Z} & \text{if } k = 1, 2; \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

which is the homology index of a repelling periodic orbit.

We have at this point established an isolating neighborhood with a non–trivial invariant set. To show that there is a periodic orbit we will use the following theorem due to C. McCord and the first two authors [8].

Theorem 4.2 *Let N be an isolating neighborhood which admits a Poincaré section. If, for some j,*

$$CH^k(\mathrm{Inv}N) \approx \begin{cases} \mathbf{Z} & \textit{if } k = j, j+1; \\ 0 & \textit{otherwise.} \end{cases}$$

then Inv$N$ *contains a periodic orbit.*

By (9), all that remains to be shown inorder to prove that $\mathrm{Inv}(N, \varphi^\epsilon)$ has a periodic orbit is to exhibit a Poincaré section for N. Recall that Σ is a *Poincaré section for* N if:

1. Σ is a local section for φ^ϵ.
2. $\Sigma \cap N$ is closed.
3. For every $x \in N$, $\varphi^\epsilon(x, (0, \infty)) \cap \Sigma \neq \emptyset$.

To construct Σ, first observe that given $0 < \delta << 1$ there are unique points y_l and y_r such that $g(p_1 - 1, y_l) = -\delta$ and $g(q_1 + 1, y_r) = \delta$. Similarly, there are unique points (x_+, y_+) and (x_-, y_-) such that $h(x_+, y_+) = \delta$ and $h(x_-, y_-) = -\delta$. Now define

$$\begin{aligned}
\mathcal{L}_1 &= \{ (p_1 - 1, y) \mid y > y_l \} \\
\mathcal{L}_2 &= \{ (q_1 + 1, y) \mid y < y_r \} \\
\mathcal{L}_- &= \{ (x, y) \mid g(x, y) = -\delta, p_1 - 1 \leq x < x_- \} \\
\mathcal{L}_+ &= \{ (x, y) \mid g(x, y) = \delta, x_+ \leq x < q_1 + 1 \}.
\end{aligned}$$

Finally, let

$$\Sigma = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_+ \cup \mathcal{L}_-.$$

Now fix $\epsilon \in (0, \bar{\epsilon}]$. For $\delta > 0$, but sufficiently small, it is easy to check that Σ satisfies properties 1 and 2 of the definition of a Poincaré section. Condition 3 is slightly more delicate. Let $(x, y) \in N$. If $g(x, y) < -\delta$, then

$$\varphi^\epsilon(0, \infty) \cap (\mathcal{L}_1 \cup \mathcal{L}_-) \neq \emptyset.$$

So assume $x > 0$ and $-\delta < g(x, y) < \delta$. Observe that for ϵ small $\dot{y} >> |\dot{x}|$. Thus, for δ sufficiently small we can again conclude that

$$\varphi^\epsilon(0, \infty) \cap (\mathcal{L}_1 \cup \mathcal{L}_-) \neq \emptyset.$$

Similar arguments take care of the remaining points in N.

Thus, Σ is Poincaré section, and hence, $\mathrm{Inv}(N, \varphi^\epsilon)$ contains a periodic orbit.

4.3 The Nagumo equations

(See [12]). Consider the Nagumo equations in one space dimension

$$\begin{aligned} u_t &= u_{xx} + f(u) - w \\ w_t &= \epsilon u. \end{aligned}$$

Here f has the form of a cubic as in Figure 7. We are looking for traveling wave solutions, i.e. solutions of the form $u(x,t) = u(x+\theta t)$ for $\theta > 0$. If we let $\xi = x + \theta t$, use $' = \frac{d}{d\xi}$, and write the travelling wave equations as a first order system we get

$$u' = v \tag{10}$$

$$v' = \theta v - f(u) + w \tag{11}$$

$$w' = \frac{\epsilon}{\theta} u. \tag{12}$$

The singular equation has $w' = 0$, so w can be considered a parameter, and we get a flow on the $u - v$ plane. In the range of w we are interested in, this flow has three critical points. The left and right roots of $f(u) = c$ give saddle points p, q in the flow, and the center root is an attractor. As shown in [12], for small positive θ, there is a $\bar{w}$ so that for w near $\bar{w}$, the phase portraits look like those in Figure 8. Here A and B are rectangles in the (u,v)-plane which we will use to construct our singular index pair. Similarly, there is a $\underline{w}$ with the phase portraits as in Figure 9. We will show that there is a periodic orbit for small ϵ. Of course, in this case the Poincaré-Bendixson theorem does not apply, so we must use the techniques described in the previous section. First, we construct a singular index pair (N, L) and show that $\mathrm{Inv}(N, \varphi^\epsilon)$ has the index of a periodic orbit, namely, that the cohomology has one generator in dimensions 1,2 and is trivial in other dimensions. Then we will show that for small ϵ there is a Poincaré section for N with respect to φ^ϵ. From these two results, it follows from Theorem 4.2 that there is a periodic orbit in N.

We construct a singular index pair as follows. To simplify the figures, we draw the left and right branches $w = f(u)$ as vertical lines. Since all the computations are done in the singular flow this has no effect on the results of the computation. To construct N, we start with the two rectangles A and B, and cross them with the w interval $[w_2, w^2]$. See Figure 10.

Now we want to add on tubes around the connecting orbits for $w = \bar{w}, \underline{w}$. To do this, for $w \in [w^1, w^2]$ follow the front face of the rectangle A forward

Figure 7: The nonlinearity for the Nagumo equations.

Figure 8: The phase portraits for $w \approx \bar{w}$.

Figure 9: The phase portraits for $w \approx \underline{w}$.

Figure 10: The vertical tubes around the critical points.

Figure 11: The horizontal tubes around the connecting orbits.

until it hits the line which contains the front face of B. Here "front" means the face which is exiting in the direction of the other critical point. We call the part of the tube in the line the *end* of the tube. This forms the tube around the $\bar{w}$ connection. For the other connecting orbit, at $w = \underline{w}$, follow the "front" face of B around until it hits the line containing the front face of A. Adding these tubes to the blocks $A \times [w_2, w^2]$ and $B \times [w_2, w^2]$ completes the construction of N. See Figure 11. Now we must determine L. The immediate exit set, N^-, consists of

$$A^{back} \times [w_2, w^2] \cup A^{front} \times [w_2, w^1) \cup B^{back} \times [w_2, w^2] \cup B^{front} \times (w_1, w^2] \cup E$$

where E is the part of the ends of the tubes which miss the A and B blocks. Taking the closure of this gives $cl(E)$ plus $A^{front} \times \{w^1\} \cup B^{front} \times \{w_1\}$. We next add the forward orbits of the closure of N^- in N. Note that nothing is added from $cl(E)$ since those points are already in the A or B tubes. The forward orbit of $A^{front} \times \{w^1\}$ adds the bottom of the $\bar{w}$ tube, and the forward orbit of $B^{front} \times \{w_1\}$ adds the top of the $\underline{w}$ tube. Putting all these pieces together gives a set which is positively invariant in N, so it remains to consider the invariant set on the boundary.

In the singular flow we have $S_0 \cap \partial N = \{(p, w_2), (p, w^2), (q, w_2), (q, w^2)\}$.

Figure 12: The set L.

The slow exit points are (p, w^2) and (q, w_2). The other two are slow entrance points. The unstable set of (p, w^2) gives a curve along the top of N from the back of A to the end of the $\bar{w}$ tube. Similarly, the unstable set of (q, w_2) gives a curve along the bottom of N from the back of B to the end of the $\underline{w}$ tube. Putting this together with the construction of the preceeding paragraph gives us L. See Figure 12. The gray region is L.

Now we want to compute the homology index. Clearly N has the homotopy type of a solid torus. L can be homotoped to two circles, one coming from A^{front}, B^{back} and the bottoms of the tubes, the other coming from A^{back}, B^{front} and the tops of the tubes. It follows that for small ϵ,

$$CH(\mathrm{Inv}(N, \varphi^\epsilon)) \approx \begin{cases} \mathbf{Z} & \text{if } k = 1, 2; \\ 0 & \text{otherwise.} \end{cases}$$

We now need to exhibit a Poincaré section for φ^ϵ in N. Let u_0 denote the middle zero of the non-linearity f, i.e. $f(u_0) = 0$. Fix a small $\delta > 0$. Define

$$\begin{aligned} \mathcal{K}_1^+ &= \{(u, v, w) \mid u = u_0,\ v > \delta,\ w_2 - 1 \leq w \leq w^2 + 1\} \\ \mathcal{K}_2^+ &= \{(u, v, w) \mid u < u_0,\ v > \delta,\ w = w_2 - 1\} \\ \mathcal{K}_3^+ &= \{(u, v, w) \mid u > u_0,\ v > \delta,\ w = w^2 + 1\} \end{aligned}$$

$$
\begin{aligned}
\mathcal{K}_4^+ &= \{(u,v,w) \mid u > 1,\ w = w^2 + 1\} \\
\mathcal{K}_1^- &= \{(u,v,w) \mid u = u_0,\ v < -\delta,\ w_2 - 1 \leq w \leq w^2 + 1\} \\
\mathcal{K}_2^- &= \{(u,v,w) \mid u > u_0,\ v < -\delta,\ w = w_2 - 1\} \\
\mathcal{K}_3^- &= \{(u,v,w) \mid u < u_0,\ v < -\delta,\ w = w^2 + 1\} \\
\mathcal{K}_4^- &= \{(u,v,w) \mid u < 0,\ w = w^2 + 1\}.
\end{aligned}
$$

Let

$$\Sigma = \bigcup_{i=1}^{4} (\mathcal{K}_i^+ \cup \mathcal{K}_i^-).$$

Arguments similar to those of the last section show that for δ sufficiently small Σ is a Poincaré section.

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