TOPOLOGICAL INVARIANTS, MULITVALUED MAPS AND COMPUTER ASSISTED PROOFS IN DYNAMICS

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ABSTRACT. We introduce the concept of inheritable properties of multivalued maps and show how this concept, when combined with interval arithmetic and representable multivalued approximations, may be used in adapting topological methods to computer assisted proofs in dynamics. As our first example we propose a simple geometric criterion for chaos. A more sophisticated example is the Conley index theory. We show how the computation of the Conley index for multivalued representable maps may be reduced to a finite computation. This reduction was used in the computer assisted proof of chaos in the Lorenz equations presented in [18].

1. INTRODUCTION

Despite the enormous achievments of the theory of dynamical systems as well as the theory of differential equations, the researcher facing the understanding of the global asymptotic dynamics of a concrete nonlinear differential equation realizes quite often that the theory is still far from being rigorously applicable to the equation of his interest. Thus, the role of mathematics in such a case reduces to providing numerical algorithms for solving the equation, whereas the theoretical achievements of the theory of dynamical systems have to be left aside. On the other hand, there is a growing number of examples exhibiting the existence of spurious solutions, i.e. solutions of some fixed qualitative behavior, which exist in a sequence of discretizations tending to the original equation but loose the behaviour in the limit (see [25]). The problems with applying the theoretical results to concrete nonlinear differential equations suggest that nonlinearity may be related to complexity of proofs, which, in concrete examples, goes beyond the human abilities. If this is really the case then the natural step would be to employ computers to help in complicated proofs. Such an approach is not a total novum, let us recall here the famous role of computers in solving the graph coloring problem (see [2, 3]). Actually, many classes of problems in discrete mathematics and algebra have been shown to be algorithmically provable or refutable

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(see [28]). Analysis did not reach that stage yet but there are signs that it will follow. In fact one of the obstacles which still distracts some people from computer assisted proofs in analysis, namely the lack of rigor caused by the presence of rounding errors in arithmetic operations on real numbers – has been overcome already in sixties by means of so called interval arithmetic (see [1]). The idea is to give up attempts to compute exact results but to get rigorous estimates of the required results instead. This makes conditions defined in terms of strong inequalities verifiable in a finite computation. Unfortunately such conditions are usually of little interest in itself. To get real applications one needs to combine interval computations with theorems which convert inequalities into equalities. A good example of such a theorem is the Darboux property used by Hastings and Troy, and then by Hassard and Zhang (see [9, 10], also comp. [11]) in their computer assisted proof of the existence of a homoclinic trajectory in the Lorenz equations. One should mention that no classical proof of this fact is known. Another example is the Banach Contraction Principle used by Koch, Schenkel and Wittwer [13] to give another proof of the existence of a solution for the Feigenbaum equation.

By its very nature, the methods of interval arithmetic may be applied to problems which either are one-dimensional or may somehow be reduced to a one-dimensional inequality condition. There are already many interesting applications of such an approach (see [6, 13, 14, 23] and references therein). Another method is to look for a bridge between the finite dynamics on finite sets, where everything is verifiable, and the infinite dynamics on continua, which appear in differential equations. Such a bridge is provided by multivalued representable maps (comp. [18, 19, 20, 27]), which arise in a natural way from numerically computed t-translation operators along the trajectories of a flow, when the computed value is surrounded by a ball of radius being the error estimate of the computed value. An important property of representable multivalued maps is that they admit finite coding, which makes them treatable by purely combinatorial methods. On the other hand, they are maps of continua with well developed continuity concepts. This causes that many set-theoretic and topological properties of such maps are verifiable in a finite computation. Of course, from the point of view of differential equations, multivalued maps do not constitute an object of interest in itself. To apply them fruitfully to single-valued dynamics, one needs verifiable properties, which are also inheritable, i.e. properties, which carry over from multivalued maps to arbitrary continuous selectors of these maps.

As one can expect, in general properties of real interest in dynamics are not inheritable. Thus also theorems converting appropriate inheritable properties into the non-inheritable properties of interest are needed. In fact, topological dynamics provides already many theorems of this kind. The aim of this paper is to show the potential of the topological methods in computer assisted proofs in dynamics along the lines described above. Of course, interval arithmetic is one of the ingredients of the proposed method but the novum lies in the global approach unifying mathematics of finite sets and continua. Moreover, it looks like the proposed topological methods offer algorithms which are essentially faster than the algorithms based on smoothness conditions. This is related to the fact that the rigorous estimation of the spatial derivative along a solution involves solving not only the original equation but also the equation in variations, which raises the dimension of the problem from n to $n + n^2$.

We begin the paper with a brief discussion of interval arithmetic in Section 2. In Section 3 we introduce representable sets as a means of discretizing the space. Then we discuss multivalued representable maps, which serve as a tool for discretizing single-valued maps. In Section 5 we define inheritable properties of multivalued maps and present some simple examples of inheritable properties. The following section contains a geometric criterion for chaotic dynamics, which is an important but elementary example of an existence criterion in dynamics based on inheritable properties. In the last two sections we show that the Conley index theory (see [4, 24, 15, 16]) is a natural source of theorems based on inheritable properties.

Throughout the paper the sets of reals, nonnegative reals, integers, nonnegative intergers, natural numbers and rationals are denoted by $\mathbb{R}, \mathbb{R}^+, \mathbb{Z}, \mathbb{Z}^+, \mathbb{N}, \mathbb{Q}$, respectively. Reals and integers supplemented by the negative and positive infinity will be denoted by $\bar{\mathbb{R}}, \bar{\mathbb{Z}}$ respectively. For any set X the notation $\mathcal{P}(X)$ will stand for the family of all subsets of X. If X is a metric space and $A \subset X$, then we denote the boundary, the interior and the closure of A respectively by $\operatorname{bd} A, \operatorname{int} A$ and $\operatorname{cl} A$. If $x \in X$ and $r > 0$ then $B(x, r)$ will denote the closed ball of center x and radius r. Similarly, if $A \subset X$ then we put $B(A, r) := \{y \in X \mid \operatorname{dist}(y, A) \leq r\}$.

2. Interval arithmetic

Let us begin with a discussion of the computer arithmetic, which, due to the presence of rounding errors, is often considered as useless from the point of view of any rigorous results. For obvious reasons, only a finite set $\hat{\mathbb{R}} \subset \mathbb{R}$ of real numbers, called *representable numbers*, can be used in computer arithmetic and the computer implementation of the four basic arithmetic operations (+,-,*,/) can be performed on the elements of this set only. The result of such an operation must be also in $\hat{\mathbb{R}}$. Thus it is usually approximate but in most present days hardware it is quaranteed that there is no other representable number between the exact result and the computed result. This makes some rigorous claims possible. To illustrate this fact with a very simple example, let us consider a (very poor) computer, which admits the following set of representable numbers

$$\hat{\mathbb{R}} = \{1.0, 1.1, 1.2, ...3.0\}$$

The question is if we can use this computer to prove that the equation

$$1.3x = 1.8$$

has a solution in the set $\mathbb{R}$ of all real numbers. Our computer produces the following table:

x	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
1.3x	1.4	1.6	1.7	1.8	2.0	2.1	2.2	2.3	2.5	2.6

The above table is not enough to claim that 1.4 is a solution, because of the presence of rounding errors. However we can claim that

$$1.3 * 1.3 < 1.8 < 1.3 * 1.5$$

This is enough to use the Bolzano Theorem (Darboux property) to conclude that a solution indeed exists.

Of course, the above example worked, because the approximate results were not involved in further computations. To solve some real problems by this method we need a rounding tracking mechanism. Such a mechanism was actually developed many years ago. It is known as the *interval arithmetic* (see [1]). Let us recall briefly the main ideas. Assume A is a subset of $\bar{\mathbb{R}}$. One defines the set of *intervals over* A by

$$\mathcal{I}(A) := \{[a, b] \mid a, b \in A, a \leq b\}$$

We treat the set of real numbers as a subset of $\mathcal{I}(\bar{\mathbb{R}})$ via the following natural embedding

$$\bar{\mathbb{R}} \ni x \to [x, x] \in \mathcal{I}(\bar{\mathbb{R}})$$

Let $\diamond \in \{+, -, *, /\}$. We extend the arithmetic operations to $I, J \in \mathcal{I}(\bar{\mathbb{R}})$ by

$$I \diamond J := \begin{cases} \{a \diamond b \mid a \in I, b \in J\} & \text{if all } a \diamond b \text{ for } a \in I, b \in J \text{ are well defined} \\ [-\infty, +\infty] & \text{otherwise} \end{cases}$$

The intervals over the set of representable numbers are called *representable intervals*. By the *representation* of a real number $x \in \mathbb{R}$ we mean any representable interval containing x. Similarly we define the *representation* of an interval $I \in \mathcal{I}(\bar{\mathbb{R}})$ as a representable interval $J \supset I$. The *optimum representation* of I is given by

$$\updownarrow I := \bigcap \{J \in \mathcal{I}(\hat{\mathbb{R}}) \mid I \subset J\}$$

We define the adjusted arithmetic operations on representable intervals $I, J \in \mathcal{I}(\hat{\mathbb{R}})$ by

$$I \mathbin{\hat{\diamond}} J := \updownarrow (I \diamond J)$$

Let us notice that the adjusted arithmetic operations on representable intervals are not asssociative in general.

Assume $Q : \mathbb{R}^p \to \mathbb{R}$ is a rational function. By an *implementation* of Q we mean a function $\mathcal{Q} : \hat{\mathbb{R}}^p \to \mathcal{I}(\hat{\mathbb{R}})$ obtained by replacing every arithmetic operation in Q by the corresponding adjusted arithmetic operation in $\mathcal{I}(\hat{\mathbb{R}})$. Note that this requires fixing some order in which the arithmetic operations will be performed. Thus a rational function may have several implementations depending on the order of evaluation of the arithmetic operations. Nevertheless one can prove easily the following proposition.

Proposition 2.1. *Assume $\mathcal{Q}$ is an implementation of a rational function Q. Then*

$$\forall (x_1, x_2, ...x_p) \in \mathbb{R}^p \; Q(x_1, x_2, ...x_p) \in \mathcal{Q}(x_1, x_2, ...x_p)$$

Interval arithmetic combined with the Darboux property provides a method to establish rigorously zeros of rational functions with the assistance of a computer. In particular it enables rigorous numerical shooting in differential equations. This approach was used recently by Hassard, Hastings, Troy and Zhang, to establish the existence of a solution homoclinic to the origin in the Lorenz system (see [9, 10, 11]). Our goal is to show how to apply in a similar way topological tools more sophisticated than the Darboux property.

3. Discretization of space.

The main idea is to build a bridge between the mathematics of finite and infinite sets. In other words we need an object, which may be treated as belonging to both worlds. We begin with discretizing the space.

Let X be a subspace of $\mathbb{R}^n$. For $\mathcal{A} \subset \mathcal{P}(X)$, $x \in X$ and $A \subset X$ denote

$$|\mathcal{A}| := \bigcup \{ a \mid a \in \mathcal{A} \}$$
$$\mathcal{A}(x) := \{ a \in \mathcal{A} \mid x \in a \}$$
$$\mathcal{A}(A) := \{ a \in \mathcal{A} \mid a \cap A \neq \emptyset \}$$

We will say that the family $\mathcal{G}$ is a *grid* in X if the following conditions are satisfied

$$\forall K \subset X \text{compact} \quad 1 \leq \operatorname{card} \mathcal{G}(K) < \infty \tag{1}$$

$$\forall a \in \mathcal{G} \quad \emptyset \neq a = \operatorname{cl}(\operatorname{int} a) \tag{2}$$

$$\forall a_1, a_2 \in \mathcal{G} \quad a_1 \neq a_2 \;\Rightarrow\; \operatorname{int} a_1 \cap \operatorname{int} a_2 = \emptyset \tag{3}$$

The grid $\mathcal{G}$ will be called convex if all elements of $\mathcal{G}$ are convex. The supremum of the diameters of the elements of $\mathcal{G}$ will be called the diameter of $\mathcal{G}$ and denoted $\operatorname{diam} \mathcal{G}$. Note that properties (2),(3) imply

$$\forall a_1, a_2 \in \mathcal{G} \quad a_1 \neq a_2 \;\Rightarrow\; a_1 \cap \operatorname{int} a_2 = \emptyset \tag{4}$$

Let $\gamma > 0$. We define the standard γ-grid in $\mathbb{R}^d$ as the collection

$$\mathcal{G}^d_\gamma := \mathcal{G}_\gamma := \{ [n_1\gamma, (n_1+1)\gamma] \times \cdots \times [n_d\gamma, (n_d+1)\gamma] \mid (n_1, \ldots n_d) \in \mathbb{Z}^d \}$$

One can easily verify the following proposition.

Proposition 3.1. *$\mathcal{G}^d_\gamma$ is a grid.*

Our next proposition summarizes basic properties of grids.

Proposition 3.2. *If $\mathcal{G}$ is a grid in X then*

$$a \in \mathcal{G},\ x \in \operatorname{int} a \ \Rightarrow\ \mathcal{G}(x) = \{\, a \,\} \tag{5}$$

$$\forall x \in X\ \exists a \in \mathcal{G}\ \exists \{\, x_n \,\} \subset \operatorname{int} a\ \ x_n \to x \tag{6}$$

$$\forall x \in X\ \exists V \text{ an open neighborhood of } x:\ y \in V \ \Rightarrow\ \mathcal{G}(y) \subset \mathcal{G}(x) \tag{7}$$

$$\forall \mathcal{A}, \mathcal{B} \subset \mathcal{G} \quad \mathcal{A} \subset \mathcal{B} \ \Leftrightarrow\ |\mathcal{A}| \subset |\mathcal{B}| \tag{8}$$

Proof: Property (5) follows immediately from (3) and property (6) from (2). To prove (7) assume the contrary. Then there exists an $x \in a \in \mathcal{G}$ and sequences $\{\, x_n \,\} \subset |\mathcal{G}|$, $\{\, a_n \,\} \subset \mathcal{G}$ such that $x_n \to x$ and $x \notin a_n$. By (1) $\mathcal{G}(\{\, x_n \,\})$ is finite. Thus passing to subsequences if necessary, we may assume that $a_n = a$ for all $n \in \mathbb{N}$. Since a is closed we get $x \in a$, a contradiction.

In order to prove (8) first observe that the implication

$$\mathcal{A} \subset \mathcal{B} \ \Rightarrow\ |\mathcal{A}| \subset |\mathcal{B}|$$

is obvious. To prove the opposite implication assume that $|\mathcal{A}| \subset |\mathcal{B}|$ and there exists an $a \in \mathcal{A} \setminus \mathcal{B}$. Choose an $x \in \operatorname{int} a$. Obviously $x \in |\mathcal{A}|$ thus also $x \in |\mathcal{B}|$, i.e. there exists a $b \in \mathcal{B}$, $b \neq a$ such that $x \in b$. Thus $b \cap \operatorname{int} a \neq \emptyset$ which contradicts (4). QED

The elements of a given grid will serve as quants of space and sets will be represented by finite families of such sets. In the quantized space something like the minimal neighborhood makes sense. It is formalized as folows. If $\mathcal{G}$ is a grid and $\mathcal{A} \subset \mathcal{G}$ then we define

$$\mathcal{A}^*_{\mathcal{G}} := \mathcal{A}^* := \{\, g \in \mathcal{G} \mid \exists a \in \mathcal{A}\ g \cap a \neq \emptyset \,\}$$

Proposition 3.3. *Assume $\mathcal{G}$ is a grid, $\mathcal{A}, \mathcal{B} \subset \mathcal{G}$ and $\mathcal{A}_\iota \subset \mathcal{G}$ for $\iota \in I$. Then*

$$\mathcal{A} \subset \mathcal{B} \ \Rightarrow\ \mathcal{A}^* \subset \mathcal{B}^* \tag{9}$$

$$\left(\bigcup \{\, \mathcal{A}_\iota \mid \iota \in I \,\}\right)^* \ =\ \bigcup \{\, \mathcal{A}^*_\iota \mid \iota \in I \,\} \tag{10}$$

$$|\mathcal{A}| \cap |\mathcal{B}| \subset |\mathcal{A} \cap \mathcal{B}^*| \tag{11}$$

Proof: Properties (9) and (10) are obvious. To prove (11) take $x \in |\mathcal{A}| \cap |\mathcal{B}|$. Then there exist $a \in \mathcal{A}$, $b \in \mathcal{B}$ such that $x \in a \cap b$. In particular $a \cap b \neq \emptyset$, i.e. $a \in \mathcal{B}^*$. Thus $x \in |\mathcal{A} \cap \mathcal{B}^*|$. QED

A compact set $Z \subset X$ will be called representable over a grid $\mathcal{G}$ if there exists a finite collection $\{\, \mathcal{Z}_i \,\}_{i=1,\dots,k}$ of subfamilies of $\mathcal{G}$ such that

$$Z = \bigcup_{i=1}^{k} \bigcap_{z_i \in \mathcal{Z}_i} z_i.$$

4. Discretization of maps.

We turn now our attention to multivalued maps as a method of discretizing single-valued maps. A map $F : X \to \mathcal{P}(Y)$, where X, Y are two arbitrary sets, will be called a *multivalued map* from X to Y and will be denoted briefly by $F : X \rightrightarrows Y$. For $A \subset X$, $B \subset Y$ we put

$$\begin{aligned} \operatorname{dom} F &:= \{\, x \in X \mid F(x) \neq \emptyset \,\} \\ F(A) &:= \bigcup \{\, F(x) \mid x \in A \,\} \\ F^{*-1}(B) &:= \{\, x \in X \mid F(x) \cap B \neq \emptyset \,\} \\ F^{-1}(B) &:= \{\, x \in \operatorname{dom} F \mid F(x) \subset B \,\} \end{aligned}$$

The set $\operatorname{dom} F$ will be called the domain of F. The sets $F(A), F^{*-1}(B), F^{-1}(B)$ will be called the image of A, the weak preimage of B and the preimage of B under F respectively. In the sequel we will use multivalued maps both, in topological setting and in purely set theoretical setting. In the first case we assume that X, Y are metric spaces and we say that F is *upper semicontinuous (usc)* if $F^{-1}(U)$ is open for any open $U \subset Y$ and F is *lower semicontinuous (lsc)* if the set $F^{*-1}(U)$ is open for any open $U \subset Y$. For $N \subset \operatorname{dom} F$ we put $\operatorname{diam}_N F := \sup\{\operatorname{diam} F(x) : x \in N\}$. The diameter of F over its domain will be called its size. If $f : X \to Y$ is a partial single-valued continuous map such that $\operatorname{dom} f = \operatorname{dom} F$ and $f(x) \in F(x)$ for every $x \in \operatorname{dom} F$ we will say that f is a selector of F.

Since in the sequel we need multivalued maps which induce maps in cohomology, we introduce the concept of star-shaped multivalued maps. Let $X \subset \mathbb{R}^p, Y \subset \mathbb{R}^q$ be two subsapces and let $F : X \rightrightarrows Y$ be a multivalued map. We will call it star-shaped if there exists a selector $f : X \to Y$ such that for any other selector $g : X \to Y$ and any $x \in \operatorname{dom} F$ the whole segment joining $f(x)$ and $g(x)$ is contained in $F(x)$. Notice that convex-valued maps which admit at least one selector are in particular star-shaped.

It is now an easy consequence of the definition of star-shaped maps that they induce maps in homology and cohomology. More precisely, if $F : X \rightrightarrows Y$ is a star-shaped multivalued map then we define

$$\begin{aligned} F_* &:= f_* : H_*(X) \to H_*(Y) \\ F^* &:= f^* : H^*(Y) \to H^*(X) \end{aligned}$$

where $f : X \to Y$ is an arbitrary selector of F. The definition is correct, because F admits at least one selector and evidently any two selectors are homotopic.

If $F : (X, X_0) \rightrightarrows (Y, Y_0)$ is a star-shaped multivalued map of pairs then in a similar way we define the maps

$$F_* : H_*(X, X_0) \to H_*(Y, Y_0)$$
$$F^* : H^*(Y, Y_0) \to H^*(X, X_0)$$

Our next step is to introduce the concept of representable multivalued maps. Assume $\mathcal{G}, \mathcal{G}'$ are grids respectively in X and Y. A multivalued map $F : X \rightrightarrows Y$ will be called representable over $\mathcal{G}, \mathcal{G}'$ if the following conditions are satisfied

$$\operatorname{dom} F \text{ is a representable set over } \mathcal{G}$$
$$\text{for every } x \in \operatorname{dom} F \text{ the set } F(x) \text{ is representable over } \mathcal{G}'$$
$$\mathcal{G}(x) = \mathcal{G}(y) \;\Rightarrow\; F(x) = F(y)$$

Let $U \subset X$ and let $f : U \to Y$ be a single valued map. We will say that a multivalued map $F : X \rightrightarrows Y$ is a *representation of f over $\mathcal{G}, \mathcal{G}'$* if F is representable over $\mathcal{G}, \mathcal{G}'$ and f is a selector of $F_{|U}$. A tool providing multivalued representations of single valued maps is based on the concept of a combinatorial multivalued map. A map $\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{G}'$ will be called a combinatorial multivalued map if $\operatorname{dom} \mathcal{F}$ is finite and for any $a \in \operatorname{dom} \mathcal{F}$ the family $\mathcal{F}(a)$ is finite. The map $\mathcal{F}$ will be called convex if for any $a \in \operatorname{dom} \mathcal{F}$ the set $|\mathcal{F}(a)|$ is convex. The map $\mathcal{F}$ will be called *a representation of f over $\mathcal{G}, \mathcal{G}'$* if $U \subset |\operatorname{dom} \mathcal{F}|$ and for every $a \in \operatorname{dom} \mathcal{F}$ and $x \in U \cap a$ we have $f(x) \in |\mathcal{F}(a)|$. We set

$$\operatorname{diam} \mathcal{F} := \max \{ \operatorname{diam} |\mathcal{F}(a)| \mid a \in \operatorname{dom} \mathcal{F} \}$$

With a combinatorial multivalued map $\mathcal{F}$ we associate the following two multivalued maps

$$\lfloor \mathcal{F} \rfloor : |\mathcal{G}| \ni x \to \bigcap \{ |\mathcal{F}(a)| \mid a \in \mathcal{G}(x) \} \in \mathcal{P}(|\mathcal{G}'|) \tag{12}$$

$$\lceil \mathcal{F} \rceil : |\mathcal{G}| \ni x \to \bigcup \{ |\mathcal{F}(a)| \mid a \in \mathcal{G}(x) \} \in \mathcal{P}(|\mathcal{G}'|) \tag{13}$$

Theorem 4.1. *The map $\lfloor \mathcal{F} \rfloor$ is lsc, the map $\lceil \mathcal{F} \rceil$ is usc and both maps are representable. Moreover, if $\mathcal{F}$ is a representation of $f : U \to X$ then also $\lfloor \mathcal{F} \rfloor$ and $\lceil \mathcal{F} \rceil$ are representations of f and* $\operatorname{diam} \lfloor \mathcal{F} \rfloor \leq \operatorname{diam} \mathcal{F}$, $\operatorname{diam} \lceil \mathcal{F} \rceil \leq 2 \operatorname{diam} \mathcal{F}$. *Additionally, if $\mathcal{F}$ is a convex representation of a single-valued map then both $\lfloor \mathcal{F} \rfloor$ and $\lceil \mathcal{F} \rceil$ are star-shaped.*

Proof: Take an $x \in X$. It follows from (7) and the definitions of $\lfloor \mathcal{F} \rfloor$ and $\lceil \mathcal{F} \rceil$ that we can choose an open neighbourhood $U(x)$ of x such that

$$x' \in U(x) \;\Rightarrow\; \lfloor \mathcal{F} \rfloor(x) \subset \lfloor \mathcal{F} \rfloor(x'),\; \lceil \mathcal{F} \rceil(x') \subset \lceil \mathcal{F} \rceil(x).$$

It follows that for any open set $V \subset Y$ we have

$$x \in \lfloor \mathcal{F} \rfloor^{*-1}(V) \;\Rightarrow\; U(x) \subset \lfloor \mathcal{F} \rfloor^{*-1}(V)$$
$$x \in \lceil \mathcal{F} \rceil^{-1}(V) \;\Rightarrow\; U(x) \subset \lceil \mathcal{F} \rceil^{-1}(V),$$

i.e. $\lfloor \mathcal{F} \rfloor$ is lsc and $\lceil \mathcal{F} \rceil$ is usc.

The rest of the assertion is an easy excercise. QED

In order to apply fruitfully representable multivalued maps to single-valued maps we need to know if single-valeud maps may be approximated by multivalued representable maps. We will say that a sequence $\{ F_n : X \rightrightarrows Y \}$ of m.v. maps uniformly converges to a single valued map $f : X \to Y$, briefly: $F_n \to f$, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for any $x \in \operatorname{dom} f$ and $n \geq N$ we have $F_n(x) \subset B(f(x), \epsilon)$. The approximation is possible in case of Lipschitz continuous maps, as the following theorems shows.

Theorem 4.2. *Assume that $\{\mathcal{G}_n\}, \{\mathcal{H}_n\}$ are two sequences of convex grids respectively in $\mathbb{R}^p$ and $\mathbb{R}^q$ with diameters going to zero. Furthermore assume that $f : \mathbb{R}^p \to \mathbb{R}^q$ is L-Lipschitz continuous and* $\operatorname{dom} f$ *is relatively compact. Then there exist sequences $\{F_n\}, \{G_n\} : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$ of representable multivalued maps such that*

(i) F_n and G_n are representations of f over $\mathcal{G}_n, \mathcal{G}'_n$
(ii) G_n is lsc , F_n is usc and they are both star-shaped
(iii) $F_n \to f$ and $G_n \to f$

Proof: Assume first that $\mathcal{G}, \mathcal{H}$ are two fixed grids in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Put $\mathcal{A} := \mathcal{G}(\operatorname{dom} f)$ and take an $\epsilon > 0$. Select finite sets of points $\{x_g\}_{g \in \mathcal{A}}$, $\{y_g\}_{g \in \mathcal{A}}$ such that $x_g \in g \cap \operatorname{dom} f$ and $\operatorname{dist}(f(x_g), y_g) < \epsilon$ and define the following combinatorial multvalued map

$$\mathcal{F} : \mathcal{G} \ni g \to \begin{cases} \mathcal{H}(B(y_g, L \operatorname{diam} \mathcal{G} + \epsilon)) & g \in \mathcal{A} \\ \emptyset & \text{otherwise} \end{cases}$$

It is straightforward to verify that $\mathcal{F}$ is a combinatorial representation of f. Take $\epsilon_n \to 0$ and let $\mathcal{F}_n$ denote the multivalued combinatorial map constructed as $\mathcal{F}$ but with respect to $\mathcal{G}_n, \mathcal{H}_n$ and ϵ_n. Put $G_n := \lfloor \mathcal{F}_n \rfloor, F_n := \lceil \mathcal{F}_n \rceil$. It is an easy excercise to verify that G_n and F_n satisfy the assertions of Theorem 4.2. QED

FIGURE 1. A multivalued representation of a single valued map f.

Figure 1 shows the idea of a construction of a multivalued representation of a one dimensional map f. Note that in practical applications several factors may contribute to the error estimate given by ϵ in the construction of a multivalued representation. For instance, if our map f is a t-translation operator along the trajectories of a differential equation with polynomial right-hand side then ϵ will be the sum of the error estimate coming from the evaluation of the vector field in interval arithmetic and the error estimate of the numerical method applied to find numerically the t-translation operator.

5. Inheritable properties of representable mv maps

As we mentioned in the introduction, the multivalued maps constitute a tool for verifying properties of single-valued maps. To make it work we need a class of properties which carry over from multivalued maps to their single-valued selectors. More precisely, assume $\mathcal{A}$ is a collection of multivalued maps and $\varphi(F)$ is a property of such maps. We will say that φ is *inheritable* if the fact that $\varphi(F)$ holds for some multivalued map $F \in \mathcal{A}$ implies that it holds for every single-valued selector of F in $\mathcal{A}$. An inheritable property φ will be called *strongly inheritable* if for any single valued map $f \in \mathcal{A}$ such that $\varphi(f)$ and for any sequence $\{F_n\} \subset \mathcal{A}$ satisfying $F_n \to f$ we have $\varphi(F_n)$ for n sufficiently large. If $\alpha(F)$ is a term then we will say that α is inheritable (strongly inheritable) if for any x the property $\alpha(F) = x$ is inheritable (strongly inheritable).

Inheritable properties are nice, because they may be proved for single-valued maps via checking them in a finite combinatorial computation for a sufficiently good multivalued representation. Unfortunately, in general they do not provide very useful information about the single valued map. Thus, the idea of applying them to solve concrete problems would be nothing else but wishful thinking if interesting theorems based on inheritable properties were not available. In the rest of the paper we want to give exapmles which show that useful theorems of this kind may be constructed. In this section we begin with very simple examples.

The simplest one is the Bolzano Theorem. Take $\mathcal{A} = \{F : \mathbb{R}^p \rightrightarrows \mathbb{R}\}$ and consider the following three properties

$$\alpha(F) \Leftrightarrow \exists x \in \mathbb{R}^p \; \forall y \in F(x) \; y > 0$$

$$\beta(F) \Leftrightarrow \exists x \in \mathbb{R}^p \; \forall y \in F(x) \; y \geq 0$$

$$\gamma(F) \Leftrightarrow \exists x \in \mathbb{R}^p \; F(x) \ni 0$$

The Bolzano Theorem may be viewed as converting strong inequalities (property α) to the existence of zeros (property γ - note that in the special case of single-valued maps this property actually turns into the existence of zeros). Property γ is not inheritable, but property α is strongly inheritable. Thus, if a single-valued map satisfies property α then a sufficiently powerful computer will be able to verify it. Property β is also inheritable but not strongly inheritable, which may cause failure in an attempt to verify it.

The above example applies only to real valued maps. Our next example, the Brouwer Fixed Point Theorem, works in any finite dimension. Assume $\mathcal{A} = \{F :$

$\mathbb{R}^p \rightrightarrows \mathbb{R}^p\}$ and let B be the unit ball in $\mathbb{R}^p$. Consider the following three properties

$$\mu(F) \Leftrightarrow F(B) \subset \operatorname{int} B$$

$$\nu(F) \Leftrightarrow F(B) \subset B$$

$$\kappa(F) \Leftrightarrow \exists x \in B\ x \in F(x)$$

The Brouwer fixed point theorem may be viewed as converting property ν into property κ, which - when restricted to single-valued maps - becomes the existence of a fixed point of the map. Again, property κ is not inheritable but property μ is inheritable. It is even better if the map satisfies property μ, which is strongly inheritable, because this property will always be verified with sufficient computational power.

6. An elementary geometric example

In this section we want to describe a geometric criterion for the existence of chaotic dynamics, in which the only assumption concerning the dynamics is inheritable. Thus the criterion is a natural tool for computer assisted proofs in dynamics and such proofs are already available (see [30, 31]). Unlike our next example, the criterion is quite elementary.

Let X be a polyhedron and X_0 its subpolyhedron.By an arc in X with endpoints in X_0 we mean a continuous map $\alpha : [0,1] \to X$, such that $\alpha(\{0,1\}) \subset X_0$ and denote it by

$$\alpha : ([0,1], \{0,1\}) \to (X, X_0).$$

We recall that two maps $f, g : (X, X_0) \to (Y, Y_0)$ are called *homotopic relative to* X_0 if there exists a continuous map $F : X \times [0,1] \to Y$ such that $F(X_0 \times [0,1]) \subset Y_0$ and $F(x,0) = f(x)$, $F(x,1) = g(x)$ for $x \in [0,1]$. It is easy to see that the homotopy relative to X_0 is an equivalence relation. Let

$$\Gamma(X, X_0) := \{ [\xi] \mid \xi : ([0,1], \{0,1\}) \to (X, X_0) \}$$

denote the family of homotopy classes of arcs in X with endpoints in X_0.

We will say that (A, A_0) is an I-pair if it is *homotopically equivalent* to the pair $([0,1], \{0,1\})$, i.e. if there exist maps

$$f : (A, A_0) \to ([0,1], \{0,1\}) \text{ and } g : ([0,1], \{0,1\}) \to (A, A_0)$$

such that gf is homotopic to identity in A relative A_0 and fg is homotopic to identity in $[0,1]$ relative $\{0,1\}$. It is easy to verify that if (A, A_0) is an I-pair then A is contractible and A_0 has exactly two arcwise connected components, which we will denote by A_0^-, A_0^+. Moreover, if $\xi_0, \xi_1 \in \Gamma(A, A_0)$ then $[\xi_0] = [\xi_1]$ iff $\xi_0(0), \xi_1(0)$ and $\xi_0(1), \xi_1(1)$ are in the same components of A_0. In particular $\Gamma(A, A_0)$ has exactly four elements $[\kappa_{i,j}]$ for $i, j \in \{-,+\}$, where $\kappa_{i,j} : ([0,1], \{0,1\}) \to (A, A_0)$ is an arc such that $\kappa_{i,j}(0) \in A_0^i$ and $\kappa_{i,j}(1) \in A_0^j$. An example of an I-pair with representants of the four elements of $\Gamma(A, A_0)$ is shown in Figure 2.

FIGURE 2. An example of an I-pair

Let $J_0 := [-2,-1]$, $J_1 := [1,2]$, $J := J_0 \cup J_1$. For $i = 0,1$ define the map

$$\mu_i : J_i \ni s \to 6s + (-1)^i 9 \in \mathbb{R}$$

and put $\mu := \mu_0 \cup \mu_1 : J \to \mathbb{R}$. Assume that $N = Z \times J$, where Z is a contractible space. Put $N^h := Z \times \{-2,-1,1,2\}$, $N^v := \mathrm{cl}(\mathrm{bd}\, N \setminus N^h)$, $N_i := Z \times J_i$, $N_i^h := Z \times \mathrm{bd}\, J_i$, $N_i^{\pm} := Z \times \mathrm{bd}^{\pm} J_i$, where $\mathrm{bd}\, J_i$ stands for the doubleton consisting of the endpoints of J_i and $\mathrm{bd}^{\pm} J_i$ denote the singletons consisting of the left and right endpoints of J_i respectively. For a point $c \in Z$ and $i = 0,1$ define a map $\mu^c : ([0,1],\{0,1\}) \to (X \setminus N^v, X \setminus N)$ by $\mu^c(s) := (c, \mu(6s-3))$.

The following theorem is a variant of a result by Zgliczyński (comp. [29]). We include its proof for sake of completness but postpone it to Appendix.

Theorem 6.1. *Assume $f : X \to X$ is a homeomorphism and there exists an I-pair (A, A_0) such that*

(i) $A \cap N^v = \emptyset$, $A_0 \cap N = \emptyset$

(ii) f maps the triple (N_i, N_i^-, N_i^+) into the triple (A, A_0^-, A_0^+).

(iii) $i : (A, A_0) \to (X \setminus N^v, X \setminus N)$ maps $[\kappa_{-,+}]$ onto $[\mu^c] \in \Gamma(X \setminus N^v, X \setminus N)$

Then for any $p \in \mathbb{N}$ and any $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{p-1} \in \{0,1\}^p$ there exists an $x \in \mathrm{Per}(N, f)$ such that $f^i(x) \in N_{\alpha_i}$ for $i \in \mathbb{Z}_p$ and $f^p(x) = x$.

FIGURE 3. An example of a map satysfying the assumptions of Theorem 6.1

An example of a two-dimensional map satisfying the assumptions of Theorem 6.1 is shown in Figure 3.

In Theorem 6.1 the only assumption concerning f is the inclusion condition on the image of N_i, N_i^-, N_i^+ under f and such a condition is obviously inheritable. Thus in order to apply the above theorem it is enough to check this condition for a multivalued representation of f which is sufficiently close to f. Evidently, this is an elementary combinatorial problem, which may be solved in a finite computation. A computer assisted proof of chaos in the Hénon map and in the Rössler equation, which is based on the above theorem may be found in [30, 31].

7. Isolating neighbourhoods and the Conley index

Our final but most important examples of inheritable properties (terms) are the concepts of the isolating neighborhood and the Conley index. Their importance comes from the fact that the Conley index has an established position as a tool in proving various existence results in dynamics ranging from the existence of stationary and periodic trajectories and heteroclinic connections to the existence of semiconjugacies onto some prescribed dynamics, including chaotic dynamics (see [4, 24, 15, 16] and references therein).

For the convenience of the reader we first recall briefly the basics of the theory as developed in [21]. Assume X is a locally compact metric space and $f : X \to X$ is a homeomorphism. Then f generates a *discrete dynamical system*

$$f : X \times \mathbb{Z} \ni (x, n) \to f(x, n) := f^n(x) \in X$$

denoted also by f.

Let $N \subset X$ be a compact set. The *invariant part* of N is given by

$$\operatorname{Inv}(N, f) := \operatorname{Inv} N := \{x \in N \mid \forall n \in \mathbb{Z}\ f^n(x) \in N\}$$

The set $S \subset X$ is called *invariant*, if $\operatorname{Inv} S = S$.

The following two properties are straightforward.

$$N \subset M \ \Rightarrow\ \operatorname{Inv} N \subset \operatorname{Inv} M \tag{14}$$

$$\operatorname{Inv}(\operatorname{Inv} N) = \operatorname{Inv} N \tag{15}$$

The set N is called an *isolating neighborhood* if any of the following two equivalent conditions is satisfied

$$\forall x \in \operatorname{bd} N\ \exists n \in \mathbb{Z}\ f(x, n) \notin N$$

$$\operatorname{Inv} N \subset \operatorname{int} N.$$

If N is an isolating neighborhood then, by (15), the set $S := \operatorname{Inv} N$ is invariant. It is called an *isolated invariant set* , becuase one can easily verify that it is the maximal invariant set contained in N

One of the fundamental and most difficult questions in dynamics is the existence of invariant sets. Isolating neighborhoods, which are relatively easier to establish, turn out to be helpful. Unfortunately, constructing an isolating neighborhhood N in itself is not sufficient, because the set $\operatorname{Inv} N$ may be, and actually often is, empty. However, the topology of N relative to the dynamics of f carries information about $\operatorname{Inv} N$ which may be used not only to show that $\operatorname{Inv} N$ is non-empty but even to deduce something about the internal structure of $\operatorname{Inv} N$. To be more precise we need the concept of an index pair.

The pair $P = (P_1, P_2)$ of compact subsets of N is called an *index pair* of S in N iff the following three conditions are satisfied

$$x \in P_i,\ f(x) \in N \ \Rightarrow\ f(x) \in P_i, \quad i = 1, 2 \tag{16}$$

$$x \in P_1,\ f(x) \notin N \ \Rightarrow\ x \in P_2 \tag{17}$$

$$\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2). \tag{18}$$

The second element in the pair is usually called the *exit set*.

It can be easily derived from the definition of the index pair that f induces a map of pairs

$$f_P : (P_1, P_2) \ni x \to f(x) \in (P_1 \cup f(P_2), P_2 \cup f(P_2))$$

and the inclusion

$$i_P : (P_1, P_2) \ni x \to x \in (P_1 \cup (P_2), P_2 \cup f(P_2)).$$

induces an isomorphism in Alexander-Spanier cohomology. The endomorphism

$$H^*(f_P) \circ H^*(i_P)^{-1} \text{ of } H^*(P),$$

where H^* denotes the Alexander-Spanier cohomology, is called the index map associated with the index pair P and denoted by I_P.

The Conley index is obtained from $(H^*(P), I_P)$ by quotienting out the generalized kernel of I_P. Thus it takes the form of a pair

$$\mathrm{Con}^*(N, f) := \mathrm{Con}^*(N) = (CH^*(N), \chi^*(N)),$$

where $CH^*(N)$ is a vector space and $\chi^*(N) : CH^*(N) \to CH^*(N)$ is an automorphism. It does not depend on the particular choice of an index pair and if N_1, N_2 are two isolating neighborhoods which isolate the same isolated invariant set S then they have the same index (see [21] for details). The fundamental property of the Conley index states that if the index is non-trivial (i.e. different from the zero vector space), then the corresponding isolated invariant set is non-empty.

As an example let us take $X := \mathbb{R}^2$, $N := \{ (x, y) \in \mathbb{R}^2 \mid |x| \leq 2,\ |y| \leq 2 \}$ and the following two maps

$$f : \mathbb{R}^2 \ni (x, y) \to (x + 1, y/2) \in \mathbb{R}^2$$
$$g : \mathbb{R}^2 \ni (x, y) \to (-2x, y/2) \in \mathbb{R}^2$$

Since both f and g are affine, it is possible and easy to find formulas for iterates of f and g and consequently an elementary computation shows that $\mathrm{Inv}(N, f) = \emptyset$, $\mathrm{Inv}(N, g) = \{ 0 \}$. Thus in both cases N is an isolating neighborhood.

Figure 4. Two isolating neighborhoods

Let us take $P = (P_1, P_2)$, $Q = (Q_1, Q_2)$, (see Figure 4), where

$$P_1 := Q_1 := N$$
$$P_2 := \{ (x, y) \in N \mid 1 \leq x \leq 2 \}$$
$$Q_2 := \{ (x, y) \in N \mid 1 \leq |x| \leq 2 \}$$

Again, an elementary computation shows that P is an index pair under f in N and Q is an index pair under g in N. The reader familiar with algebraic topology will also easily compute that $\mathrm{Con}(N, f) = \{ 0 \}$, $\mathrm{Con}(N, g) = (\mathbb{R}, -\mathrm{id})$. But even without knowledge of algebraic topology the reader shouldn't hesitate to believe that in case of g, contrary to f, it is not possible to construct an index pair with contractible exit set and that this is related to the fact that $\operatorname{Inv}(N, g) \neq \emptyset$.

We will show that both the isolating neighborhood and the index pair constitute inheritable terms (actually, one can show in case of the usc mv maps that the Conley index theory extends to a certain subclass of such maps – comp. [12]) and the Conley index becomes an inheritable property. We need to explain first how these concepts may be extended to multivalued dynamical systems. Assume $F : X \times \mathbb{Z} \rightrightarrows X$ is a multivalued map. We will say that F is a *multivalued discrete dynamical system* (mvdds) on the space X if the following conditons are satisfied

$$F(x, 0) = x$$

$$y \in F(x, n) \;\Leftrightarrow\; x \in F(y, -n)$$

$$nm \geq 0 \;\Rightarrow\; F(F(x, n), m) = F(x, n + m).$$

Assume $N \subset X$. A function $\sigma : \mathbb{Z} \to N$ satisfying $\sigma(0) = x$ and $\sigma(n+1) \in F(\sigma(n))$ will be called a solution through x in N. We put

$$\operatorname{Inv}(N) := \operatorname{Inv}(N, F) := \{x \in N \mid \exists \sigma : \mathbb{Z} \to N \text{ a solution through } x\}.$$

The following Proposition follows immediately from the definition of a solution and the definition of a selector.

Proposition 7.1. *If f is a selector of F then*

$$\operatorname{Inv}(N, f) \subset \operatorname{Inv}(N, F).$$

The set N is said to be an isolating neighbourhood for F if

$$B_{\operatorname{diam}_N F}(\operatorname{Inv} N) \subset \operatorname{int} N,$$

where $\operatorname{diam}_N F$ is the maximal diameter of the values of F in N. It is said to be a *pseudo isolating neighborhood* for F if

$$\forall x \in \operatorname{bd} N \; \exists n \in \mathbb{Z} \; F(x, n) \cap N = \emptyset.$$

Theorem 7.2. *Both the pseudo isolating neighborhood and the isolating neighborhood are inheritable terms. The isolating neighborhood is also a strongly inheritable term.*

Proof: Let N be a pseudo isolating neighborhood for F and let f be a selector of F. Let $x \in \operatorname{Inv}(N, f)$. Then $f^n(x) \in N \cap F(x, n)$ for all $n \in \mathbb{Z}$. It follows that $x \notin \operatorname{bd} N$, thus $x \in \operatorname{int} N$.

Assume in turn that N is an isolating neighborhood for F. Then

$$\operatorname{Inv}(N, f) \subset \operatorname{Inv}(N, F) \subset B_{\operatorname{diam}_N F}(\operatorname{Inv}(N, F)) \subset \operatorname{int} N.$$

QED

Let N be an isolating neighbourhood for F. A pair $P = (P_1, P_2)$ of compact subsets $P_2 \subset P_1 \subset N$ is called an *index pair* if the following conditions are satisfied:

$$F(P_i) \cap N \subset P_i,\ i = 1, 2; \tag{19}$$

$$F(P_1 \backslash P_2) \subset N; \tag{20}$$

$$\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2) \tag{21}$$

Proposition 7.3. *If P is an index pair for N then the inclusion*

$$i_P : F(P) \subset T(P) := (P_1 \cup F(P_2), P_2 \cup F(P_2))$$

induces an isomorphisms in the Alexander-Spanier cohomology.

In order to define the index map in the multivalued case we need mv maps which induce maps in cohomology. Thus, from now on, we restrict our attention to star-shaped maps.

Let $F_{P,T(P)} : P \rightrightarrows T(P)$ be the restriction of F to domain P and codomain $T(P)$. The endomorphism $I_P := H^*\left(F_{P,T(P)}\right) \circ H^*(i_P)^{-1}$ of $H^*(P)$ is called the *index map* associated with the index pair P.

Theorem 7.4. *Assume N is an isolating neighborhood with respect to a mvds F and $P = (P_1, P_2)$ is an index pair in N. Then for any single-valued selector f of F the pair P is also an index pair for f. Moreover, the index maps induced by F and f coincide.*

Proof: Properties (16),(17) are immediate. Property (18) follows from Proposition 7.1. Homomorphism $H^*(F_{P,T(P)})$ and $H^*(f_{P,T(P)})$ coincide by the definition of cohomology induced maps by star-shaped maps. Thus also the index maps induced by F and f coincide. QED

Theorem 7.4 in particular means that one can extend the Conley index to discrete multivalued dynamical systems generated by star-shaped maps by taking as the index the Conley index of any its selector.

Corollary 7.5. *The index pair, the index map and consequently also the Conley index are strongly inheritable terms.*

The fact that the Conley index is an inheritable term does not immediately mean that every condition formulated in terms of the Conley index is inheritable. To see this take $X = S^1 = \mathbb{R}/\mathbb{Z}$ and

$$f : X \ni [x] \to [x(2-x)] \in X$$
$$F : X \ni [x] \to \{\,[y] \mid y \in [f(x) - 1/8, f(x) + 1/8]\,\} \in \mathcal{P}(X)$$
$$A := \{\,[x] \mid x \in [0, 1/2]\,\},$$

where $[x]$, for $x \in \mathbb{R}$, denotes the equivalence class of x in $\mathbb{R}/\mathbb{Z}$ and $[a, b]$, for $a, b \in \mathbb{R}$, denotes the closed interval in $\mathbb{R}$. Then $F(A) = S^1$, thus $F(A)$ is an isolating neighborhood for F. However $[0]$, a fixed point of f, is easily seen to belong to $\operatorname{bd} f(A)$. Thus $f(A)$ is not an isolating neighborhood under f, which means that "$F(A)$ is an isolating neighborhood for F" is not an inheritable property.

This example shows that not every theorem in terms of the Conley index automatically provides a tool for computer assisted proofs.

Sometimes a reformulation helps. The following theorem is a special case of [17], Theorem 2.3.

Theorem 7.6. *Let $N = N_0 \cup N_1$ be an isolating neighborhood under f where N_0 and N_1 are disjoint compact sets. For $k, l = 0, 1$, let $M_{kl} = N_k \cap f(N_l)$. Let $N_{lk} := M_{kk} \cup M_{kl} \cup M_{ll}$. Assume that*

$$\operatorname{Con}^n(N_k, f) = \begin{cases} (\mathbb{Q}, \operatorname{id}) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

and that $\chi^*(N_{lk}, f)$ is not conjugate to $\chi^*(N_k, f) \oplus \chi^*(N_l, f)$. Then, there exists a $d \in \mathbb{N}$ and a continuous surjection $\rho : \operatorname{Inv}(N, f) \to \Sigma_2$ such that

$$\rho \circ f^d = \sigma \circ \rho$$

where $\sigma : \Sigma_2 \to \Sigma_2$ is the full shift dynamics on two symbols.

The assumptions of this theorem are not inheritable, because the isolating neighborhoods in the assumption are defined in terms of the map itself, like in the preceding example. However, Theorem 7.6 implies the following theorem, in which all assumptions are inheritable.

Theorem 7.7. *Assume M_{ij} for $i, j \in \{0, 1\}$ are disjoint compact sets such that $N := M_{00} \cup M_{01} \cup M_{10} \cup M_{11}$ is an isolating neighborhood under f. Furthermore assume that $N_i := M_{i0} \cup M_{i1}$ satisfy*

$$N \cap f(N_i) \subset M_{0i} \cup M_{1i} \tag{22}$$

$$\operatorname{Con}^n(N_k, f) = \begin{cases} (\mathbb{Q}, \operatorname{id}) & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

and $\chi^*(N_{lk}, f)$ is not conjugate to $\chi^*(N_k, f) \oplus \chi^*(N_l, f)$. Then, there exists a $d \in \mathbb{N}$ and a continuous surjection $\rho : \operatorname{Inv}(N, f) \to \Sigma_2$ such that

$$\rho \circ f^d = \sigma \circ \rho$$

where $\sigma : \Sigma_2 \to \Sigma_2$ is the full shift dynamics on two symbols.

Proof: Put $\bar{M}_{ij} := N_i \cap f(N_j)$ and $\bar{N}_{ij} := \bar{M}_{00} \cap \bar{M}_{ij} \cap \bar{M}_{11}$. It is sufficient to show that $\operatorname{Inv} \bar{N}_{ij} = \operatorname{Inv} N_{ij}$ for $i \neq j$, because then also the assumptons of Theorem 7.6 are satisfied. We have

$$\bar{M}_{ij} = N_i \cap f(N_j) \subset (M_{i0} \cup M_{i1}) \cap (M_{0j} \cup M_{1j}) = M_{ij}$$

Thus $\bar{N}_{ij} \subset N_{ij}$ and by (14) $\operatorname{Inv} \bar{N}_{ij} \subset \operatorname{Inv} N_{ij}$. We will show that $\operatorname{Inv} N_{ij} \subset \bar{N}_{ij}$. Assume the contrary. Then there exists an $x \in \operatorname{Inv} N_{ij} \setminus \bar{N}_{ij}$. In particular $x = f(u)$ for some $u \in N_{ij} \subset N$. We have either $x \in N_i$ or $x \in M_{jj}$. In the first case we get $u \notin N_i$ and $u \notin N_j$, a contradiction. In the other case we get $u \in N_i$, thus $x \in M_{0i} \cup M_{1i}$ but also $x \in M_{jj}$, a contradiction again, because $i \neq j$. Now (14) and (15) show that $\operatorname{Inv} \bar{N}_{ij} = \operatorname{Inv} N_{ij}$. QED

8. Reduction to combinatorial problems

In the preceding section we showed that the Conley index theory also leads to inheritable properties. However, unlike our previous example, it is not so evident that the computation of the Conley index for a representable multivalued map reduces to a finite combinatorial computation. Although a representable multivalued map admits a finite coding, an infinite number of iterates of F is involved in the definition of the isolating neighborhood, which makes the problem delicate. More precisely, what is needed are algorithmic methods of verifying if a given compact set is an isolating neighborhood with respect to a given representable multivalued dynamical system and finding an index pair and consequently the Conley index. In this section we address this question restricted to a relatively simple special case of the isolating neighborhood, the so called isloating blocks. The answer in the general case is also positive but since the case is much more complicated it will be treated seperately in [22].

The task becomes relatively simple if we reduce our attention to isolating blocks. We say that N is an *isolating block* if

$$B_{\operatorname{diam}_N F}(F^{*-1}(N) \cap N \cap \operatorname{cl} F(N)) \subset \operatorname{int} N \tag{24}$$

It is straightforward to prove the following

Proposition 8.1. *We have*

$$\operatorname{Inv} N \subset F^{*-1}(N) \cap N \cap F(N)$$

Corollary 8.2. *If N is an isolating block then N is an isolating neighborhood.*

To be able to construct an index pair we need to restrict our attention to lsc and usc maps only. We will treat each case seperately.

We begin with the following

Theorem 8.3. *Assume F is usc and N is an isolating block for F. Define*

$$N^- := N \backslash F^{-1}(\operatorname{int} N).$$

Then (N, N^-) is an index pair for F in N.

Proof: First observe that since F is usc, N^- is closed. Obviously N satisfies (19). In order to prove (19) for N^- take $x \in N^-$ and assume $y \in F(x) \cap N \setminus N^-$. Then $y \in F^{-1}(\operatorname{int} N)$, i.e. $F(y) \subset \operatorname{int} N \subset N$. Hence we have

$$y \in F^{-1}(N) \cap N \cap F(N) \subset \operatorname{cl} F^{*-1}(N) \cap N \cap \operatorname{cl} F(N)$$

Now, $x \in N^-$ implies $F(x) \not\subset \operatorname{int} N$. Let $z \in F(x) \setminus \operatorname{int} N$. Since both y and z are in $F(x)$, we have from (24)

$$z \in B(y, \operatorname{diam}_N F) \subset B(F^{*-1}(N) \cap N \cap F(N), \operatorname{diam}_N F) \subset \operatorname{int} N,$$

a contradiction.

Assume in turn that $x \in N$ and $F(x) \setminus N \neq \emptyset$. Then $F(x) \not\subset \operatorname{int} N$ and consequently $x \in N^-$, which proves (20)

Finally observe that $\operatorname{int}(N \setminus N^-) = \operatorname{int} N \cap F^{-1}(\operatorname{int} N)$. It follows from Proposition 8.1 that $\operatorname{Inv} N \subset \operatorname{int} N$. In order to prove (21) we need to show that $\operatorname{Inv} N \subset F^{-1}(\operatorname{int} N)$. For this end take $x \in \operatorname{Inv} N$. Then $F(x) \cap \operatorname{Inv} N \neq \emptyset$, i.e. $F(x) \subset B(\operatorname{Inv} N, \operatorname{diam}_N F) \subset \operatorname{int} N$. This completes the proof. QED

Proposition 8.4. *Assume $\mathcal{G}, \mathcal{G}'$ are grids in X, $\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{G}'$ is a combinatorial multivalued map and $\mathcal{N} \subset \mathcal{G}$. Then*

$$\lceil \mathcal{F} \rceil (|\mathcal{N}|) = |\mathcal{F}(\mathcal{N}^*)| \tag{25}$$

$$\operatorname{cl} \lceil \mathcal{F} \rceil^{*-1} (|\mathcal{N}|) = |\mathcal{F}^{*-1}(\mathcal{N}^*)| \tag{26}$$

$$\operatorname{cl}(|\mathcal{N}| \setminus \lceil \mathcal{F} \rceil^{-1} (|\mathcal{N}|) = |\mathcal{N} \setminus \mathcal{F}^{-1}(\mathcal{N})| \tag{27}$$

Proof: Property (25) follows from the following sequence of mutually equivalent conditions.

$$\begin{array}{c}
y \in \lceil \mathcal{F} \rceil (|\mathcal{N}|) \\
\exists x \; x \in |\mathcal{N}|, y \in \lceil \mathcal{F} \rceil (x) \\
\exists x \exists a \; x \in |\mathcal{N}|, a \in \mathcal{G}(x0, y \in |\mathcal{F}(a)| \\
\exists x \exists a \exists g \; x \in g, g \in \mathcal{N}, x \in a, y \in |\mathcal{F}(a)| \\
\exists a \exists g \; g \in \mathcal{N}, g \cap a \neq \emptyset, y \in |\mathcal{F}(a)| \\
\exists a \; a \in \mathcal{N}^* y \in |\mathcal{F}(a)| \\
y \in |\mathcal{F}(\mathcal{N}^*)|
\end{array}$$

Similarly, property (26) follows from the following sequence of mutually equivalent conditions.

$$\begin{gathered}
x \in \lceil\mathcal{F}\rceil^{*-1}(|\mathcal{N}|)\\
\lceil\mathcal{F}\rceil(x) \cap |\mathcal{N}| \neq \emptyset\\
\exists y\; y \in \lceil\mathcal{F}\rceil(x) \cap |\mathcal{N}|\\
\exists y \exists u \exists v \exists a\; x \in u \in \mathcal{N}, y \in v \in \mathcal{N}, v \in \mathcal{F}(u), y \in a \in \mathcal{N}\\
\exists u \exists v \exists a\; x \in u, v \cap a \neq \emptyset, v \in \mathcal{F}(u)\\
\exists u \exists v\; x \in u, v \in \mathcal{N}^* \cap \mathcal{F}(u)\\
\exists u\; x \in u, u \in \mathcal{F}^{*-1}(\mathcal{N}^*)\\
x \in |\mathcal{F}^{*-1}(\mathcal{N}^*)|
\end{gathered}$$

In order to prove that

$$\mathrm{cl}(|\mathcal{N}| \setminus \lceil\mathcal{F}\rceil^{-1}(|\mathcal{N}|) \subset |\mathcal{N} \setminus \mathcal{F}^{-1}(\mathcal{N})|$$

it is enough to prove that

$$|\mathcal{N}| \setminus \lceil\mathcal{F}\rceil^{-1}(|\mathcal{N}| \subset |\mathcal{N} \setminus \mathcal{F}^{-1}(\mathcal{N})|,$$

because the set $|\mathcal{N} \setminus \mathcal{F}^{-1}(\mathcal{N})|$ is compact. Thus take $x \in |\mathcal{N}| \setminus \lceil\mathcal{F}\rceil^{-1}(|\mathcal{N}|)$. Then there exists an $a \in \mathcal{N}$ such that $x \in a$ and $|\mathcal{F}(a)| \not\subset |\mathcal{N}|$. Then, by (8) $\mathcal{F}(a) \not\subset \mathcal{N}$, i.e. $a \in \mathcal{N} \setminus \mathcal{F}^{-1}(\mathcal{N})$ and consequently $x \in |\mathcal{N} \setminus \mathcal{F}^{-1}(\mathcal{N})|$.

To prove the opposite inclusion take $x \in |\mathcal{N} \setminus \mathcal{F}^{-1}(\mathcal{N})|$. Choose an $a \in \mathcal{N} \setminus \mathcal{F}^{-1}(\mathcal{N})$ such that $x \in a$. By (6) we can choose $\{x_n\} \subset \operatorname{int} a$, a sequence of points such that $x_n \to x$. Since then $\mathcal{G}(x_n) = \{\, a \,\}$, we have $\lceil\mathcal{F}\rceil(x_n) = |\mathcal{F}(a)|$ and $|\mathcal{F}(a)| \not\subset |\mathcal{N}|$ by (8) . It follows that $x_n \in |\mathcal{N}| \setminus \lceil\mathcal{F}\rceil^{-1}(|\mathcal{N}|)$ and $x \in \mathrm{cl}(|\mathcal{N}| \setminus \lceil\mathcal{F}\rceil^{-1}(|\mathcal{N}|))$.

QED

We have now the following

Theorem 8.5. . *Assume $\mathcal{G}$ is a grid in X and $\mathcal{F} : \mathcal{G} \rightrightarrows \mathcal{G}$ is a combinatorial multivalued map. If $\mathcal{N} = \operatorname{dom} \mathcal{F}$ is such that*

$$B_{2\,\mathrm{diam}\,\mathcal{F}}(|\mathcal{F}^{*-1}(\mathcal{N}^*) \cap \mathcal{F}(\mathcal{N}^*)^*|, \mathrm{bd}\,|\mathcal{N}|) \subset \operatorname{int} |\mathcal{N}| \tag{28}$$

then $|\mathcal{N}|$ is an isolating block for $\lceil\mathcal{F}\rceil$ and $(|\mathcal{N}|, |\mathcal{N} \setminus \mathcal{F}^{-1}(\mathcal{N})|)$ is an index pair for $\lceil\mathcal{F}\rceil$ in $|\mathcal{N}|$.

Proof: From Proposition 8.4, property (11) and Theorem 4.1 we obtain

(29)

$$B_{\mathrm{diam}\,\lceil\mathcal{F}\rceil}(\lceil\mathcal{F}\rceil^{*-1}(|\mathcal{N}| \cap |\mathcal{N}| \cap \lceil\mathcal{F}\rceil(|\mathcal{N}|)) \subset B_{2\,\mathrm{diam}\,\mathcal{F}}(|\mathcal{F}^{*-1}(\mathcal{N}^*) \cap \mathcal{F}(\mathcal{N}^*)^*|) \subset \operatorname{int} |\mathcal{N}|$$

Thus $|\mathcal{N}|$ is an isolating neighborhood for $\lceil\mathcal{F}\rceil$. The rest of the assertion follows from Theorem 8.3 and property (27). QED

Although the theory of lower semicontinuous multivalued maps is considered in general to be less satisfactory than the case of the upper semicontinuous maps, from the point of view of computer assisted proofs it seems to be an attractive alternative, because in the constructions given by (13) one has to double the diameter of numerically computed maps just to extend the map to the points in the boundary of the grid. This means that it is harder to find an isolating neighborhood. In the other case we do not need to double the diameter.

The theorem below shows how an index pair may be found in case of an isolating block and a lower semicontinuous mvds.

Theorem 8.6. *Assume F is lsc and N is an isolating block for F. Define*

$$N^- := N \setminus \operatorname{int}_N(N \cap F^{-1}(N)).$$

Then (N, N^-) is an index pair for F in N.

Proof: Obviously N^- is closed and N satisfies (19). In order to prove (19) for N^- take $x \in N^-$ and assume that $y \in F(x) \cap N \setminus N^-$. Then $y \in \operatorname{int}_N(N \cap F^{-1}(N))$ and there exists a sequence $\{x_n\} \subset N$ such that $x_n \to x$ and $x_n \notin F^{-1}(N)$. Select $z_n \in F(x_n) \setminus N$ and take $\epsilon > 0$ such that

$$B(F^{*-1}(N) \cap N \cap F(N), \operatorname{diam}_N F + \epsilon) \subset \operatorname{int} N.$$

Lower semicontinuity of F implies that there exists an $n_0 \in \mathbb{N}$ such that $F(x) \subset B(F(x_{n_0}), \epsilon)$. Since $y \in F(x)$ there exists a $z \in F(x_{n_0})$ such that $\operatorname{dist}(y, z) \leq \epsilon$. Hence

$$\operatorname{dist}(y, z_{n_0}) \ \leq \ \operatorname{dist}(y, z) + \operatorname{dist}(z, z_{n_0}) \ \leq \ \epsilon + \operatorname{diam}_N F.$$

Since $y \in F^{*-1}(N) \cap N \cap F(N)$, it follows that

$$z_{n_0} \in B(F^{*-1}(N) \cap N \cap F(N), \operatorname{diam}_N F + \epsilon) \subset \operatorname{int} N,$$

a contradiction. This proves (19) for N^-.

Assume in turn that $x \in N$ and $F(x) \setminus N \neq \emptyset$. Then $x \notin F^{-1}(N)$, i.e. $x \in N^-$, which proves (20). Finally observe that $\operatorname{int}(N \setminus N^-) = \operatorname{int} N \cap \operatorname{int} F^{-1}(N)$. In order to prove (21) it is sufficient to show that $\operatorname{Inv} N \subset \operatorname{int} F^{-1}(N)$, because $\operatorname{Inv} N \subset \operatorname{int} N$ by Proposition 8.1. If $\operatorname{Inv} N \not\subset \operatorname{int} F^{-1}(N)$ then there exists a sequence $\{x_n\} \subset X$ such that $x_n \to x \in \operatorname{Inv} N$ and $F(x_n) \not\subset N$. Select $z_n \in F(x_n) \setminus N$ and take an $\epsilon > 0$ and an $n_0 \in \mathbb{N}$ as in the proof of (19). Since $x \in \operatorname{Inv} N$ we can take a $y \in F(x) \cap \operatorname{Inv} N$. Then $y \in F(x) \subset B(F(x_{n_0}), \epsilon)$, hence there exists a $z \in F(x_{n_0})$ such that $\operatorname{dist}(y, z) \leq \epsilon$. Arguing as in the proof of (19) we find that $z_{n_0} \in \operatorname{int} N$, a contradiction. QED

Proposition 8.7. *Assume $\mathcal{G},\mathcal{G}'$ are grids in X, $\mathcal{F}:\mathcal{G}\rightrightarrows\mathcal{G}'$ is a multivalued map and $\mathcal{N}\subset\mathcal{G}$. Then*

$$\lfloor\mathcal{F}\rfloor(|\mathcal{N}|)=|\mathcal{F}(\mathcal{N})| \tag{30}$$

$$\operatorname{cl}\lfloor\mathcal{F}\rfloor^{*-1}(|\mathcal{N}|)=|\mathcal{F}^{*-1}(\mathcal{N})| \tag{31}$$

$$\operatorname{cl}(|\mathcal{N}|\setminus\lfloor\mathcal{F}\rfloor^{-1}(|\mathcal{N}|)=|\mathcal{N}\setminus\mathcal{F}^{-1}(\mathcal{N})| \tag{32}$$

Proof: Let $y\in\lfloor\mathcal{F}\rfloor(x)$ for some $x\in|\mathcal{N}|$. Let $a\in\mathcal{A}(x)\cap\mathcal{N}$. Then $y\in|\mathcal{F}(a)|\subset|\mathcal{F}(\mathcal{N})|$. Thus $\lfloor\mathcal{F}\rfloor(|\mathcal{N}|)\subset|\mathcal{F}(\mathcal{N})|$. On the contrary, if $y\in|\mathcal{F}(\mathcal{N})|$ then $y\in|\mathcal{F}(a)|$ for some $a\in\mathcal{N}$. Choose $x\in\operatorname{int}a$. Then $x\in|\mathcal{N}|$ and $\mathcal{A}(x)=\{a\}$. Consequently $\lfloor\mathcal{F}\rfloor(x)=|\mathcal{F}(a)|$, i.e. $y\in\lfloor\mathcal{F}\rfloor(|\mathcal{N}|)$. This proves (30).

If $x\in\lfloor\mathcal{F}\rfloor^{*-1}(\mathcal{N})$ then there exists an $a\in\mathcal{N}$ such that $x\in a$ and $\mathcal{F}(a)\cap\mathcal{N}\neq\emptyset$. Thus $a\in\mathcal{F}^{*-1}(\mathcal{N})$ and $x\in|\mathcal{F}^{*-1}(\mathcal{N})|$. This shows that $\lfloor\mathcal{F}\rfloor^{*-1}(|\mathcal{N}|)\subset|\mathcal{F}^{*-1}(\mathcal{N})|$ and since $|\mathcal{F}^{*-1}(\mathcal{N})|$ is closed also $\operatorname{cl}\lfloor\mathcal{F}\rfloor^{*-1}(|\mathcal{N}|)\subset|\mathcal{F}^{*-1}(\mathcal{N})|$. To prove the opposite inclusion take $x\in a\in\mathcal{A}$ such that $\mathcal{F}(a)\cap\mathcal{N}\neq\emptyset$. By (6) there exists a sequence $\{x_n\}\subset\operatorname{int}a$ of points such that $x_n\to x$. By (5) $\lfloor\mathcal{F}\rfloor(x_n)\cap|\mathcal{N}|=|\mathcal{F}(a)|\cap|\mathcal{N}|\subset|\mathcal{F}(a)\cap\mathcal{N}|\neq\emptyset$, i.e. $x_n\in\lfloor\mathcal{F}\rfloor^{*-1}(|\mathcal{N}|)$ and consequently $x\in\operatorname{cl}\lfloor\mathcal{F}\rfloor^{*-1}(|\mathcal{N}|)$. This proves (31).

Assume in turn that $x\in|\mathcal{N}|\setminus\lfloor\mathcal{F}\rfloor^{-1}(|\mathcal{N}|)$. Thus there exists an $a\in\mathcal{A}$ such that $x\in a$ and $\mathcal{F}(a)\not\subset\mathcal{N}$. Thus $a\in\mathcal{N}\setminus\mathcal{F}^{-1}(\mathcal{N})$ and $x\in|\mathcal{N}\setminus\mathcal{F}^{-1}(\mathcal{N})|$. This proves that the left-hand side of (32) is contained in the right-hand side. To prove the opposite inclusion take $x\in a\in\mathcal{N}$ such that $\mathcal{F}(a)\not\subset\mathcal{N}$. Take $a'\in\mathcal{F}(a)\setminus\mathcal{N}$ and $x'\in\operatorname{int}a'$. Then, by (5), $x'\in|\mathcal{F}(a)|\setminus|\mathcal{N}|$, i.e. $|\mathcal{F}(a)|\not\subset|\mathcal{N}|$. Now let $\{x_n\}\subset\operatorname{int}a$ be a sequence of points such that $x_n\to x$. Then $\lfloor\mathcal{F}\rfloor(x_n)=|\mathcal{F}(a)|\not\subset|\mathcal{N}|$, i.e. $x_n\in|\mathcal{N}|\setminus\lfloor\mathcal{F}\rfloor^{-1}(|\mathcal{N}|)$ and consequently $x\in\operatorname{cl}(|\mathcal{N}|\setminus\lfloor\mathcal{F}\rfloor^{-1}(|\mathcal{N}|))$. This proves (32). QED

Now we are able to prove

Theorem 8.8. . *Assume $\mathcal{G}$ is a grid in X and $\mathcal{F}:\mathcal{G}\rightrightarrows\mathcal{G}$ is a combinatorial multivalued map. If $\mathcal{N}=\operatorname{dom}\mathcal{F}$ is such that*

$$B_{\operatorname{diam}\mathcal{F}}(|\mathcal{F}^{*-1}(\mathcal{N})\cap\mathcal{F}(\mathcal{N})^*|,\operatorname{bd}|\mathcal{N}|)\subset\operatorname{int}|\mathcal{N}| \tag{33}$$

then $|\mathcal{N}|$ is an isolating block for $\lfloor\mathcal{F}\rfloor$ and $(|\mathcal{N}|,|\mathcal{N}\setminus\mathcal{F}^{-1}(\mathcal{N})|)$ is an index pair for $\lfloor\mathcal{F}\rfloor$ in $|\mathcal{N}|$.

Proof: The proof is analogous to the proof of Theorem 8.5 but is based on Theorem 8.6 and Proposition 8.7 QED

It is a straightforward task to write algorithms verifying conditions (28) and (33) and constructing the associated index pair for a given $\mathcal{N}$ and $\mathcal{F}$, because both $\mathcal{N}$ and $\mathcal{F}$ are finite objects. Thus Theorems 8.5 and 8.8 show that finding the Conley index of a representable multivalued dynamical system induced by a combinatorial representable multivalued map reduces to a combinatorial computation, at least in case of an isolating block. The fundamental question what is the complexity of finding the Conley index remains an important open problem. However, regardless of what the complexity is, it is low enough to admit concrete applications. A computer assisted proof of chaos in the Lorenz equations based on Theorem 8.8 with a prototype algorithm verifying condition (33) is presented in [18] and [19]. More advanced techniques will be discussed in [20].

9. Appendix

The main tool needed in the proof of Theorem 6.1 is the fixed point index (see [5]). It is an integer associated with a map and a set, which – whenever non-zero – implies that the map has a fixed point in the given set. To be more concrete, assume that X is a nice topological space like a polyhedron (the general assumption is X is a metric ANR), $f : X \to X$ is a continuous map and $N \subset X$ is a compact subset. The set of fixed points of f is

$$\operatorname{Fix}(f) := \{\, x \in X \mid f(x) = x \,\}$$

We say that N isolates its fixed points if $N \cap \operatorname{Fix} f \subset \operatorname{int} N$, i.e. there are no fixed points of f on the boundary of N. If this is the case then the fixed point index $i(N, f)$ is defined. Its main features are

$$i(N,f) \neq 0 \;\Rightarrow\; N \cap \operatorname{Fix} f \neq \emptyset \tag{34}$$

$$N \cap \operatorname{Fix} f = N' \cap \operatorname{Fix} f \subset \operatorname{int} N \cap \operatorname{int} N' \;\Rightarrow\; i(N,f) = i(N,f') \tag{35}$$

$$N \cap \operatorname{Fix} f_\lambda \subset \operatorname{int} N,\ \lambda \in [0,1] \;\Rightarrow\; i(N,f_\lambda) = \text{const} \tag{36}$$

In the last property it is assumed that $\{f_\lambda\}_{\lambda\in[0,1]}$ is a family of continuous map continuously depending on λ.

Periodic points of f are just fixed points of some iterate of f. Thus it is not surprising that the fixed point index may be also used to detect periodic trajectories. Let us define

$$\operatorname{Per}(N,f) := \{\, x \in \operatorname{Fix}(f^n) \mid n \in \mathbb{N},\ f^i(x) \in N,\ i = 0, 1, \dots n \,\}$$

We will say that N isolates its periodic points if $\operatorname{Per}(N,f) \subset \operatorname{int} N$.

For $p \in \mathbb{N}$ let $\mathbb{Z}_p = \{\, 0, 1, \dots, p \,\}$ denote the cyclic group of order p. Assume now that N is a union of two its disjoint closed subsets N_0 and N_1. Take $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{p-1}) \in \{0,1\}^{\mathbb{Z}^p}$. We are interested in the existence of a periodic point x of f which follows the pattern prescribed by α, i.e. $f^i(x) \in N_{\alpha_i}$ for $i \in \mathbb{Z}^p$. Thus for $M \subset N$ define

$$G^*_\alpha(M,f) := \bigcap \{\, f^{-i}(M \cap N_{\alpha_i}) \mid i \in \mathbb{Z}_p \,\}$$

and

$$G_\alpha(M,f) := \bigcup \{\, G^*_\beta(M,f) \mid \beta \in [\alpha] \,\}.$$

where

$$[\alpha] = \{\, \beta \in \{\, 0, 1 \,\}^{\mathbb{Z}_p} \mid \exists k \in \mathbb{Z}_p \; \beta_{i+k} = \alpha_i \text{ for } i \in \mathbb{Z}_p \,\}$$

Proposition 9.1. *If N isolates periodic points, then also $G_\alpha(N, f)$ isolates periodic points.*

Proof: Assume $G_\alpha(N, f)$ does not isolate its periodic points. Then there exists an $x \in \operatorname{bd} G_\alpha(N, f) \cap \operatorname{Per}(G_\alpha(N, f), f)$. Since the sets $G^*_\beta(N, f)$ for $\beta \in [\alpha]$ are easily seen to be pairwise disjoint, we see that $x \in \operatorname{bd} f^{-i}(N_{\beta_i})$ for some $i \in \mathbb{Z}_p$ and consequently $f^i(x) \in \operatorname{bd} N_{\beta_i} \subset \operatorname{bd} N$. Obviously also $f^i(x) \in \operatorname{Per}(N, f)$, a contradiction. QED

Assume now that a family of continuous maps $\{\, f_\lambda : X \to X \,\}_{\lambda \in [0,1]}$ continuously depending on λ is given. The following proposition is an easy excercise (comp. [21]).

Proposition 9.2. *Assume that N isolates periodic points for all $f_\lambda, \lambda \in \Lambda$. Then the following maps are upper semicontinuous*

$\lambda \to \operatorname{Fix} f_\lambda$
$\lambda \to G_\alpha(N, f_\lambda)$
$\lambda \to G_\alpha(N, f_\lambda) \cap \operatorname{Fix} f_\lambda$

Proposition 9.1 in particular means that $G_\alpha(N, f)$ isolates the fixed points of f^p, thus the fixed point index $i(G_\alpha(N, f), f^p)$ is well defined. We will show the following theorem

Theorem 9.3. *If $\{f_\lambda : X \to X\}_{\lambda \in [0,1]}$ is a family of continuous maps continuously depending on λ and N isolates periodic points of f_λ then*

$$i(G_\alpha(N, f_0^p)) = i(G_\alpha(N, f_1^p)).$$

The theorem follows immediately from properties (35) and (36) and the following lemma.

Lemma 9.4. *Assume that N isolates periodic points of f_λ for $\lambda \in \Lambda$ and $M \subset \operatorname{int} N$ is a compact set such that $N \cap \operatorname{Fix} f^p_{\lambda_0} \subset \operatorname{int} N$ for some $\lambda_0 \in [0, 1]$. Then there exists Δ, a neighborhood of λ_0 such that for every $\lambda \in \Delta$*

$$G_\alpha(N, f_\lambda) \cap \operatorname{Fix} f^p_\lambda = G_\alpha(M, f_{\lambda_0}) \cap \operatorname{Fix} f^p_\lambda$$

Proof: Since $G_\alpha(M, f_{\lambda_0}) \subset G_\alpha(\operatorname{int} N, f_{\lambda_0}) \subset G_\alpha(N, f_{\lambda_0})$, it follows from the upper semicontinuity of $\lambda \to G_\alpha(N, f_\lambda)$ that the right-hand side is contained in the left-hand side for λ close to λ_0. Similarly, since $G_\alpha(N, f_{\lambda_0}) \cap \operatorname{Fix} f^p_{\lambda_0} \subset G_\alpha(\operatorname{int} M, f_{\alpha_0}) \subset \operatorname{int} G_\alpha(N, f_{\lambda_0})$, it follows from the upper semicontinuity of $\lambda \to G_\alpha(N, f_\lambda) \cap \operatorname{Fix} f^p_\lambda$ that the opposite inclusion holds true for λ close to λ_0. QED

Proof of Theorem 6.1. Define $(D^v, D) := (X \setminus N^v, X \setminus N)$. For $i = 1, 2$ define the arc $\zeta_i : ([0, 1], \{0, 1\}) \to (A, A_0)$ by $\zeta_i(s) := f(c, s - \frac{1+(-1)^i 3}{2})$ for $s \in [0, 1]$. Assumption (ii) implies that ζ_1, ζ_2 are well defined and $[\zeta_i] = [\kappa_{-,+}]$. Thus, by assumption (iii) there exists a homotopy $u_t^i : ([0,1],\{0,1\} \to (D^v, D)$ such that $u_0^i = \zeta_i$, $u_1^i = \mu^c$. Define $w_t : J \to Z$ by

$$w_t(s) := \begin{cases} u_t^0(s+2) & s \in J_0 \\ u_t^1(s-1) & s \in J_1 \end{cases}.$$

Let $r_t : Z \to Z$ be a homotopy joining id_T and c_T. Define homotopies

$$\bar{h}_t(z,s) := f(r_t(z), s)$$

$$\bar{\bar{h}}_t(z,s) := w_t(s)$$

and let $\{ h_t \}$ be the composition of these homotopies. Assumption (ii) and the definition of w_t imply that also

$$h_t : (N, N^h) \to (D^v, D).$$

We will show that N isolates periodic points of h_t. Indeed, if $x \in \operatorname{bd} N$ then either $x \in N^h$ or $x \in N^v$. If $x \in N^h$ then $h_t(x) \notin N$. If $x \in N^v$ then $x \notin h_t(N)$. Thus in both cases x cannot be a periodic point in N.. We have $h_0 = f$, $h_1 : (x,s) \to (c, \mu(s))$. Now, let $p \in N$ and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{p-1}) \in \{0,1\}^p$. Theorem 9.3 implies that $i(G_\alpha(N,f), f^p) = i(G_\alpha(N,h_1), h_1^p)$. Since h_1 is a product of a constant map and an explicitely given affine map, the computation of $i(G_\alpha(N,h_1), h_1^p)$ is a straightforward task and the index turns out to be nontrivial. Thus also $i(G_\alpha(N,f), f^p) \neq 0$ by property (36) of the fixed poin index and the conclusion follows from proeprty (34) of the fixed point index. QED

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