# The Conley Index and Rigorous Numerics

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## 1 Introduction.

The Conley index is a topological invariant of isolated invariant sets, invented by C. Conley and his students in early 70's. The theory originated from the paper on existence of smooth isolating blocks by Conley and Easton [5]. The first concept of the index appeared in a conference announcement by Conley [3], in which main ideas of the homotopy and cohomological index were presented. The basic idea of the Conley index is to generalize the Morse index so as to make it defined also in the degenerate case and to give it features similar to the fixed point index (homotopy property, additivity property etc.).

The foundations of the theory, in a locally compact setting, were established in the papers of R. Churchill [2] and J. Montgomery [55] (the cohomological index), in the book of C. Conley [4] (the homotopy index and the generalized Morse index) and in the paper of H. Kurland [33] (the Morse index). The generalization of the theory to the case of a non locally compact space was given by K. Rybakowski in [78, 79, 80]. This admitted direct

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applications of the theory to partial differential equations and functional differential equations. The more recent developments of the theory can be found in [1, 16, 18, 19, 20, 21, 22, 31, 34, 35, 39, 40, 41, 43, 57, 58, 59, 60, 61, 62, 65, 66, 77, 88, 89, 94, 95].

The Conley index theory found numerous applications in the theory of ordinary and partial differential equations and in dynamical systems (see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 23, 24, 28, 29, 30, 42, 44, 45, 46, 47, 48, 72, 73, 74, 75, 76, 81, 82, 83, 84, 85, 86]). In recent years the theory also showed its importance in the field of rigorous numerics. The goal of rigorous numerics is to perform numerical analysis in such a way that the results of computations provide proofs of interesting results. Examples of such an approach may be found in [17, 25, 26, 27, 32, 36, 37, 50, 51, 52, 53, 70, 71, 93].

The aim of this review paper is to present how the Conley index theory may be fruitfully applied to rigorus numerics.

Throughout the paper the sets of reals, nonnegative reals, integers, nonnegative intergers, natural numbers and rationals are denoted by $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$, $\mathbb{Q}$, respectively. For any set $X$ the notation $\mathcal{P}(X)$ will stand for the family of all subsets of $X$. If $X$ is a metric space with metric $d$ and $A \subset X$, then we denote the boundary, the interior and the closure of $A$ respectively by $\operatorname{bd} A, \operatorname{int} A$ and $\operatorname{cl} A$. If $x \in X$ and $r > 0$ then $B(x, r)$ will denote the closed ball of center $x$ and radius $r$. Similarly, if $A \subset X$ then we put $B(A, r) := \{y \in X \mid d(y, A) \leq r\}$. The Alexander–Spanier cohomology of a pair of topological spaces $(X, X_0)$ other a fixed ring $R$ will be denoted by $H^*(H, X_0; R)$ or briefly by $H^*(X, X_0)$.

## 2 The Conley index for flows.

The Conley index is assigned to isolated invariant sets of dynamical systems. Assume $X$ is a locally compact metric space and $\pi : X \times \mathbb{R} \to X$ is a flow on $X$. For a compact $N \subset X$ we define the invariant part of $N$ by

$$\operatorname{Inv} N := \{x \in N \mid \forall t \in \mathbb{R}\; \pi(x, t) \in N\}.$$

We say that $S$ is invariant if $S = \operatorname{Inv} S$. We say that $S$ is an isolated invariant set if there exists a compact set $N$ such that $S = \operatorname{Inv} N$ and $S \subset \operatorname{int} N$. The set $N$ is then called an isolating neighborhood.

There are several ways to define the Conley index, depending on the tools used in the construction (the cohomological, homotopy, shape Conley index). Thus it is convenient to give an axiomatic definition of the Conley index.

Assume $\mathcal{E}$ is a category and $\mathcal{C}$ is a certain class of pairs $(S, \pi)$, where $S$ is an isolated invariant set of the flow $\pi$.

By a Conley index on $\mathcal{C}$ we mean a map $C : \mathcal{C} \to \mathcal{E}$, which satisfies the following four properties:

1. **Waewski property**: If $C(S, \pi)$ is not the zero object of $\mathcal{C}$ then $S$ is non-empty.

2. **Homotopy property**: Assume $\pi_s, s \in [0, 1]$, is a family of dynamical systems continuously depending on $s$ and $N \subset X$. If $(\text{Inv } N, \pi_s) \in \mathcal{C}$ for all $s \in [0, 1]$ then $C(\text{Inv } N, \pi_s)$ does not depend on s.

3. **Additivity property**: If $(S_1, \pi), (S_2, \pi) \in \mathcal{C}$ then $(S_1 \cup S_2, \pi) \in \mathcal{C}$ and the Conley index of $S_1 \cup S_2$ is the co-product or product of the indices of $S_1$ and $S_2$.

4. **Normalization property**: Assume $X$ and $\pi$ are smooth and $x_0$ is a hyperbolic stationary point of $\pi$. Then $(\{x_0\}, \pi) \in \mathcal{C}$ and $C(\{x_0\}, \pi)$ is non-zero.

The above axioms were chosen among the most often used properties of the Conley index for flows. It is not known to the author if the axioms are independant and if they uniquely characterize the Conley index.

The definition of the homotopy (cohomological) Conley index for flows rests on the concept of a Waewski set. A compact set $W \subset X$ is called a Waewski set if

$$W^- := \{x \in W | \; \exists \epsilon > 0 \; : \phi(x, t) \notin W \text{ for } 0 < t < \epsilon\}$$

is closed. In order to define the Conley index one first proves the following two theorems.

**Theorem 2.1** *Every isolating neighborhood $N$ contains a Ważewski set $W$ such that $Inv\, N \subset W$.*

**Theorem 2.2** *The homotopy type of the quotient space $W/W^-$ (the Alexander-Spanier cohomology $H^*(W, W^-)$) does not depend on the index pair but only on the isolated invariant set $S$.*

This common value is taken as the homotopy (cohomological) Conley index.

Though the homotopy index is more general and easier to introduce (all necessary definitions from homotopy theory are elementary), the cohomological Conley index is easier to apply, because the cohomology is easier to compute than the homotopy type. Thus we will restrict our attention to the cohomology Conley index in the sequel. The cohomology Conley index takes the form of a graded modulus if the cohomology coefficients are in a ring or a vector space if the coefficients are in a field.

The definition of the Conley index for semidynamical systems is similar (see [78, 79, 80]) but conceptually more complicated, so we omit the details.

# 3 The Conley index for maps.

The first step towards applying the Conley index in rigorous numerics is constructing it for discrete dynamical systems. This is because, when a differential equation is investigated numerically, a discrete numerical scheme approximating the equation is iterated. Recall that the discrete dynamical system on $X$ is a group $\pi = \{\pi_t\}_{t\in\mathbb{Z}}$ of homeomorphisms of $X$. Unlike the flow, a discrete dynamical system is generated by a single element, namely $\pi_1$ (or $\pi_{-1}$). Hence it is often identified with its generator. In other words, if $f : X \to X$ is a homeomorphism then we can think of $f$ as a discrete dynamical system $\pi_f = \{f^n\}_{n\in\mathbb{Z}}$ given by the iterates of $f$.

Already Ch. Conley [4] observed that the construction of an analog of the index for discrete dynamical systems would be of interest but the first construction (in smooth setting) is due to Robbin and Salamon [77]. Parts of the theory (the notions of isolating neighborhood and isolated invariant set) can be carried over directly to the discrete case. However, the notion of Ważewski set does not make sense in the discrete case. This is because the trajectories may jump over the boundary. Thus it is necessary to extend the concept of exit set in such a way that it is not necessarily a subset of boundary. This is done via so called index pairs.

**Definition 3.1** *The pair $P = (P_1, P_2)$ of compact subsets of $N$ will be called an index pair of $S$ in $N$ iff the following three conditions are satisfied*

$$x \in P_i,\ f(x) \in N \ \Rightarrow\ f(x) \in P_i, \quad i = 1, 2$$

$$x \in P_1,\ f(x) \notin N \ \Rightarrow\ x \in P_2$$
$$Inv\, N \subset int(P_1 \backslash P_2).$$

Like in the case of a flow (Theorem 2.1) one can prove

**Theorem 3.2** *Every isolating neighbourhood admits an index pair $P$.*

However there is no direct analog of Theorem 2.2. The cohomology of index pair does depend on its choice. The proof fails, because the fundamental tool in the proof of Theorem 2.2, i.e. the homotopy built along trajectories of the flow, does not make any sense in the discrete case. In this respect entirely new ideas are needed. We follow here the author's approach presented in [58].

Let $P = (P_1, P_2)$ be an index pair in $N$. It can be easily derived from the definition of the index pair that $f$ induces a map of pairs

$$f_P : (P_1, P_2) \ni x \to f(x) \in (P_1 \cup f(P_2), P_2 \cup f(P_2))$$

and the inclusion

$$i_P : (P_1, P_2) \ni x \to x \in (P_1 \cup (P_2), P_2 \cup f(P_2)).$$

induces an isomorphism in Alexander-Spanier cohomology.

**Definition 3.3** *The endomorphism*

$$H^*(f_P) \circ H^*(i_P)^{-1} \ \textit{of}\ H^*(P),$$

*is called the (cohomological) index map associated with the index pair $P$ and denoted by $I_P$.*

The index map contains information which is essential in the construction of the discrete Conley index. This information is not important in the continuous case, because it is then trivial in the sense that the index map associated with the time-one translation map of a flow is always the identity.

In order to make use of the extra information we need some definitions. Together with each category $\mathcal{E}$ we consider the category of endomorphisms of $\mathcal{E}$ denoted by $\mathrm{Endo}(\mathcal{E})$. The objects of $\mathrm{Endo}(\mathcal{E})$ are pairs $(A, a)$, where $A \in \mathcal{E}$ and $a \in \mathcal{E}(A, A)$ is a distinguished endomorphism of $A$. The set of morphisms from $(A,a) \in \mathcal{E}$ to $(B,b) \in \mathcal{E}$ is the subset of $\mathcal{E}(A,B)$ consisting of exactly those morphisms $\varphi \in \mathcal{E}(A,B)$ for which $b\varphi = \varphi a$. We write $\varphi : (A,a) \to (B,b)$ to denote that $\varphi$ is a morphism from $(A,a)$ to $(B,b)$ in $\mathrm{Endo}(\mathcal{E})$. We define the category of automorphisms of $\mathcal{E}$ as the full subcategory of $\mathrm{Endo}(\mathcal{E})$ consisting of pairs $(A,a) \in \mathrm{Endo}(\mathcal{E})$ such that $a \in \mathcal{E}(A,A)$ is an automorphism, i.e., both an endomorphism and an isomorphism in $\mathcal{E}$. The category of automorphisms of $\mathcal{E}$ will be denoted by $\mathrm{Auto}(\mathcal{E})$. There is a functorial embedding

$$\mathcal{E} \ni A \to (A, \underset{A}{\mathrm{id}} \in \mathrm{Auto}(\mathcal{E}))\,,$$

$$\mathcal{E}(A,B) \ni \varphi \to \varphi \in \mathrm{Auto}(\mathcal{E})(A,B)\,,$$

hence we can consider the category $\mathcal{E}$ as a subcategory of $\mathrm{Auto}(\mathcal{E})$. Assume $\mathcal{C}$ is a full subcategory of $\mathrm{Endo}(\mathcal{E})$ and $L : \mathcal{C} \to \mathrm{Auto}(\mathcal{E})$ is a functor. Let $(A,a) \in \mathcal{C}$. Then $L(A,a)$ is an object of $\mathrm{Auto}(\mathcal{E})$. Let $a'$ denote the automorphism distinguished in $L(A,a)$. Obviously $a : (A,a) \to (A,a)$ is a morphism in $\mathrm{Endo}(\mathcal{E})$ and since $\mathcal{C}$ is a full subcategory of $\mathrm{Endo}(\mathcal{E})$ it is also a morphism in $\mathcal{C}$. Hence $L(a)$ is defined and it is a morphism from $L(A,a)$ to $L(A,a)$ in $\mathrm{Auto}(\mathcal{E})$. However, it need not be $L(a) = a'$ in general. $L(a)$ need not be even an isomorphism in $\mathrm{Auto}(\mathcal{E})$. We say that $L : \mathcal{C} \to \mathrm{Auto}(\mathcal{E})$ is normal, if for each $(A,a) \in \mathcal{C}$ the morphism $L(a)$ is equal to the automorphism distinguished in $L(A,a)$.

There are many ways to define normal functors and actually one can prove the existence of a universal normal functor (see Szymczak [91]). One of the simplest constructions applies to the category of finitely dimensional graded vector spaces $\mathcal{V}$.

Let $(F,f) \in \mathrm{Endo}(\mathcal{V})$. Define the generalized kernel of $f$ as

$$\mathrm{gker}(f) := \bigcup \{ f^{-n}(0) | n \in \mathbb{N} \}.$$

Since $f(\mathrm{gker}(f)) \subset \mathrm{gker}(f)$, we have an induced monomorphism

$$f' : F/\,\mathrm{gker}(f) \ni [x] \to [f(x)] \in F/\,\mathrm{gker}(f).$$

Put

$$L(F,f) := (F/\,\mathrm{gker}(f), f').$$

It is easy to verify that $f'$ is in fact an isomorphism. Thus $L(F, f) \in \mathcal{E}I$. One can easily extend the above definition to morphisms, so that we indeed obtain a covariant functor

$$L : \mathcal{E}E \to \mathcal{E}I.$$

We will call it the Leray functor. One easily verifies that the Leray functor is normal. Other examples of normal functors may be obtained in terms of the direct and inverse limits (see [62]).

Let $\mathcal{M}$ denote the category of graded moduli over the ring $R$ and let $L : \mathrm{Endo}(\mathcal{M}) \to \mathrm{Auto}(\mathcal{M})$ be a normal functor. The analog of Theorem 2.2 in the discrete case is the following theorem.

**Theorem 3.4** *Assume $f : X \to X$ is a homeomorphism and $S$ is an isolated invariant set with respect to $\pi_f$. Then $L(H^*(P), I_P)$ does not depend on the index pair but only on the isolated invariant set $S$.*

The above theorem allows us to define the Conley index in the discrete case as $L(H^*(P), I_P)$. Let us emphasize that in the discrete case the Conley index has the form of a pair

$$\mathrm{Con}^*(S) = (CH^*(S), \chi^*(S)),$$

where $CH^*(S)$ is a graded vector space and $\chi^*(S) : CH^*(S) \to CH^*(S)$ is a graded automorphism.

As we stated above, if $f$ is the time-one-map of a flow and $S$ is an isolated invariant set, then $I_P$ is an identity. Consequently

$$\mathrm{Con}^*(S, \pi_f) = L(H^*(P), \mathrm{id}) = (H^*(P), \mathrm{id}) = H^*(P).$$

Hence we have the following

**Theorem 3.5** *(see [61]) The cohomological Conley index of an isolated invariant set of a flow coincides with the corresponding cohomological Conley index of the time-one-map of this flow.*

There is also

**Theorem 3.6** *(see [61]). $S$ is an isolated invariant set with respect to a flow iff it is an isolated invariant set with respect to the time-one translation of the flow.*

Hence our index can be considered as a generalization of the cohomological Conley index for flows.

# 4 A horseshoe example

In this section we want to show how the Conley index for maps may be computed directly from its definition. For the convenience of the reader we first briefly recall basic definitions and notation concerning the Alexander–Spanier cohomology.

For a locally compact metric space $X$ and $n \in \mathbb{N}$, let $\Gamma^n(X)$ denote the $\mathbb{Z}$-module of all functions $\varphi : X^{n+1} \to \mathbb{Z}$. The coboundary homomorphism $\delta : \Gamma^n(X) \to \Gamma^{n+1}(X)$ is defined by the formula

$$\delta\varphi(x_0, \dots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \varphi(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}).$$

Since $\delta\delta = 0$, $\Gamma^*(X) := (\Gamma^n(X), \delta)$ is a cochain complex over $\mathbb{Z}$. For $\varphi \in \Gamma^n(X)$ define the support of $\varphi$ by $|\varphi| :=$

$$\{x \in X \mid \forall\, V \subset X \text{ open } x \in V \Rightarrow \exists\, x_0, \dots, x_n \in V \text{ s.t. } \varphi(x_0, \dots, x_n) \neq 0\}.$$

Then, $\Gamma_0^n(X) := \{\varphi \in \Gamma^n(X) \mid |\varphi| = \emptyset\}$ is a submodule of $\Gamma^n(X)$ and $\Gamma_0^*(X) := (\Gamma_0^n(X), \delta)$ is a cochain subcomplex of $\Gamma^*(X)$. The quotient cochain complex $\Gamma^*(X)/\Gamma_0^*(X)$ is denoted by $\bar{\Gamma}^*(X)$. If $u \in \bar{\Gamma}^*(X)$ then $|\varphi|$ does not depend on $\varphi$ for all representations $\varphi \in u$. Hence one can set $|u| := |\varphi|$, where $\varphi \in u$.

If $Y$ is another topological space and $g : X \to Y$ is continuous then there is an induced cochain map $g^\# : \Gamma^*(Y) \to \Gamma^*(X)$ defined by the formula

$$(g^\#\varphi)(x_0, \dots, x_n) = \varphi(g(x_0), \dots, g(x_n)).$$

Since $g^\#(\Gamma_0^*(Y)) \subset \Gamma_0^*(X)$, there is also an induced map $g^\# : \bar{\Gamma}^*(Y) \to \bar{\Gamma}^*(X)$.

If $i : X_0 \hookrightarrow X$ is an inclusion, then $\bar{\Gamma}^*(X, X_0) := \ker\, i^\#$ is a cochain subcomplex of $\bar{\Gamma}^*(X)$. The Alexander–Spanier cohomology of the pair $(X, X_0)$ is defined as the homology of the cochain subcomplex $\bar{\Gamma}^*(X, X_0)$ and denoted by $\bar{H}^*(X, X_0)$. If $(Y, Y_0)$ is another topological pair and $\kappa : \bar{\Gamma}^*(Y, Y_0) \to \bar{\Gamma}^*(X, X_0)$ is a cochain map then it induces the map $\kappa^* : \bar{H}^*(Y, Y_0) \to \bar{H}^*(X, X_0)$. In particular if $g : (X, X_0) \to (Y, Y_0)$ is a continuous map then one has a map $(g^\#)^* : \bar{H}^*(Y, Y_0) \to \bar{H}^*(X, X_0)$, which for the sake of simplicity will be denoted by $g^*$.

**Proposition 4.1** *(see [38], Sect. 1.1) Assume* $\varphi \in \Gamma^*(X)$*. Then*

$$[\varphi] \neq 0 \quad \Rightarrow \quad |\varphi| \neq \emptyset.$$

**Proposition 4.2** *.see [50], Prop. 3.5 Assume* $X_0, X_1 \subset X$ *are closed subsets of* $X$ *such that*

$$X = int\, X_0 \cup int\, X_1.$$

*Then*

$$\bar{H}^*(X, X_0 \cap X_1) \cong \bar{H}^*(X, X_0) \oplus \bar{H}^*(X, X_1)$$

.

One can show (see [56]) that every isolating neighborhood admits a regular index pair, i.e. an index pair $P$ which additionally satisfies the following condition

$$\mathrm{cl}(P_1 \backslash P_2) \cap \mathrm{cl}(f(P_2) \backslash P_1) = \emptyset,$$

In case of regular index pairs one can define the index map also on the chain level. For this end we need the following proposition

**Proposition 4.3** *. If* $P$ *is a regular index pair then* $i_P : (P_1, P_2) \to (P_1 \cup f(P_2), P_2 \cup f(P_2))$ *induces an isomorphism* $i^\# : \bar{\Gamma}^*(P_1 \cup f(P_2), P_2 \cup f(P_2)) \to \bar{\Gamma}^*(P_1, P_2)$ *and*

$$(i_P^{\#,p})^{-1}([\varphi]) = [\lambda_P(\varphi)]$$

*where* $\lambda_P : \Gamma^*(P_1, P_2) \to \Gamma^*(P_1 \cup f(P_2), P_2 \cup f(P_2))$ *is defined by*

$$\lambda_P(\varphi)(x_0, \ldots, x_p) = \begin{cases} \varphi(x_0, \ldots, x_p) & \text{if } x_0, \ldots, x_p \in P_1 \\ 0 & \text{otherwise.} \end{cases}$$

**Proof:** The proposition follows easily from the proof of Lemma 4 in [90], Sect. 6.4. QED

The cochain map $f^\# \circ (i^\#)^{-1}$ is called the cochain index map and is denoted by $f_P^\#$. Obviously

$$f_P^* = (f_P^\#)^*.$$

Let $J_0 := [-12, -6]$, $J_1 := [6, 12]$, $J := [-12, 12]$. For $i = 0, 1$ define the maps

$$\mu_i : J_i \ni s \to 6s + (-1)^i 54 \in \mathbb{R}$$

$$\nu_i : [-18, 18] \ni s \to (s - (-1)^i 54)/6 \in J_i$$

Put $N := J \times J$, $N_i := J \times J_i$ and for $i = 0, 1$ define maps

$$f_i : N_i \ni (x, y) \to (\mu_i(x), \nu_i(y)) \in \mathbb{R}^2.$$

Now extend the map $f_0 \cup f_1$ to a homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ in such a way that $f(N \setminus (N_0 \cup N_1)) \cap N = \emptyset$. An elementary computation shows that

$$f^{-1}(N) \cap N \cap f(N) = \bigcup_{i,j=0,1} K_i \times K_j,$$

where $K_0 := [-11, -7]$, $K_1 := [7, 11]$. In particular we obtain

$$\operatorname{Inv}(N, f) \subset f^{-1}(N) \cap N \cap f(N) \subset \operatorname{int} N,$$

which shows that $N$ is an isolating neighborhood for $f$.

One easily verifies that $(N, E)$, where $E := [-12, -11] \cup [-7, 7] \cup [11, 12]$ is an index pair for $f$ in $N$. Taking $E_0 := [-12, -11] \cup [-7, 12]$ and $E_1 := [-12, 7] \cup [11, 12]$ we get $E = E_0 \cap E_1$ and by Proposition 4.2 we obtain

$$H^*(N, E) = H^*(N, E_0) \oplus H^*(N, E_1).$$

Using the exact sequence of a pair (see [38], Sect.8.2) we conclude that

$$H^i(N, E_0) = H^i(N, E_1) = \begin{cases} \mathbb{Z} & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$H^i(N, E) = \begin{cases} \mathbb{Z}^2 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let us turn now our attention to computing the index map. For $s \in J$ define the map

$$\varphi_s : N^2 \ni ((x_0, y_0), (x_1, y_1)) \to \begin{cases} 1 & \text{if } x_0 < s, x_1 \geq s, \\ -1 & \text{if } x_0 \geq s, x_1 < s, \\ 0 & \text{otherwise.} \end{cases}$$

and let $u_s := \varphi_s + \Gamma^1_0(N)$. One easily finds that $|u_s| = \{ (x, s) \mid x \in J \}$. In particular we get

$$u_s \in \begin{cases} \bar{\Gamma}^1(N, E_0) & \text{for } s \in L_0 \\ \bar{\Gamma}^1(N, E_1) & \text{for } s \in L_1. \end{cases},$$

where $L_0 := (-11, -7)$, $L_1 := (7, 11)$. Let $s_1, s_2 \in L_0, s_1 < s_2$. Define

$$\psi_{s_1,s_2} : N \ni (x, y) \to \begin{cases} 1 & \text{if } s_2 > x \geq s_1, \\ 0 & \text{otherwise.} \end{cases}$$

and put $v_{s_1,s_2} := \psi_{s_1,s_2} + \Gamma_0^0(N)$. An elementary computation shows that $\delta v = u_{s_1} - u_{s_2}$ and $\delta u_s = 0$, i.e. $u_{s_1}, u_{s_2}$ are homologous co-cycles in $\bar{\Gamma}^1(N, E_0)$. An analogous argument shows that $u_{s_1}, u_{s_2}$ are homologous co-cycles in $\bar{\Gamma}^1(N, E_1)$ for $s_1, s_2 \in L_1, s_1 < s_2$. Let $\alpha_i := [u_s]$, where $s \in L_i$.

Since by Proposition 4.1 $u_s \neq 0$ and obviously $u_s$ is not a non-trivial multiple of another function, we conclude that $\alpha_i$ generates $H^1(N, E_i)$ for $i = 0, 1$.

Now we easily verify that

$$f_P^*(\alpha_0) = [f_P^\#(u_{-9})] = [u_{7.5} + u_{-10.5}] = [u_{7.5}] + [u_{-10.5}] = [u_9] + [u_{-9}] = \alpha_0 + \alpha_1$$

and

$$f_P^*(\alpha_1) = [f_P^\#(u_9)] = [u_{-7.5} + u_{10.5}] = [u_{-7.5}] + [u_{10.5}] = [u_{-9}] + [u_9] = \alpha_0 + \alpha_1.$$

Thus the matrix of $f_P^*$ in the basis $\alpha_0, \alpha_1$ is

$$A := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Since $A^n = 2^{n-1}A$, we see that $\operatorname{gker} A = \ker A$ and we easily conclude that in case of coefficients in the field of rationals and the Leray functor

$$\operatorname{Con}^n(N, f) = \begin{cases} (\mathbb{Q}, 2\,\mathrm{id}) & \text{if n=1,} \\ 0 & \text{otherwise.} \end{cases}$$

# 5 Properties of the Conley index for maps

Similarly to the Conley index for flows the Conley index for maps has several properties which faciliate its computation. Some of them are analogous to the properites of the Conley index for flows but some are specific to the discrete case. In this section we would like to summarize them.

**Theorem 5.1** ***Ważewski property:*** *If $N$ is an isolating neighbourhood with respect to $\pi_f$ and $Con(Inv\,N, \pi_f) \neq 0$ then $Inv\,N$ is non-empty.*

**Theorem 5.2** ***Stability property:*** *If $N$ is an isolating neighbourhood with respect to $\pi_f$ then it is an isolating neighbourhood with respect to $\pi_g$ for $g$ sufficiently close to $f$ in the compact-open topology and the Conley indexes with respect to $f$ and $g$ coincide.*

**Theorem 5.3** ***Homotopy property:*** *Assume $\pi_s$, $s \in [0,1]$, is a family of dynamical systems on $X$ continuously depending on $s$. If $N \subset X$ is an isolating neighbourhood with respect to every dynamical system $\pi_s$ then $Con(Inv\, N, \pi_s)$ does not depend on $s$.*

**Theorem 5.4** ***Additivity property:*** *If $N, N_1, N_2$ are isoltaing neighbourhoods such that $Inv\, N = Inv\, N_1 \cup Inv\, N_2$, $Inv\, N_1 \cap Inv\, N_2 = \emptyset$ then*

$$Con(Inv\, N, \pi_f) = Con(Inv\, N_1, \pi_f) \times Con(Inv\, N_2, \pi_f)$$

**Theorem 5.5** ***Normalization property:*** *The whole space $X$ is an isolating neighbourhood and $Con(X, \pi_f) = (H^*(X), f^*)$.*

**Theorem 5.6** ***Commutativity property*** *(see [61]). Assume $f = \psi\phi, g = \phi\psi$, where $\phi : X \to Y, \psi : Y \to X$ are continuous,. If $S \subset X$ is an isolated invariant set with respect to $\pi_f$ then $\phi(S)$ is an isolated invariant set with respect to $\pi_g$ and $Con(S, \pi_f) = Con(\phi(S), \pi_g)$.*

**Theorem 5.7** ***Relation to fixed point index*** *(see [58]). Assume $X$ is a compact ANR and $S$ is an isolated invariant set of $\pi_f$. Let $Con(S, \pi_f) = (E, e) \in \mathcal{E}I$. Then $E$ is of finite type and $\Lambda(e)$, the Lefschetz number of $e$ is exactly the fixed point index of $f$ in a neighbourhood of $S$.*

Let $x_0$ be a hyperbolic fixed point of $f$. Let $k$ denote the number of eigenvalues of Df($x_0$) with modulus greater than one (counted with multiplicity). Let $l$ denote the number of real eigenvalues of Df($x_0$) which are less than -1. Then the pair $(k, l)$ will be called the Morse index of $x_0$.

**Theorem 5.8** *Assume $x_0$ is a hyperbolic fixed point of a $C^1-$diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$. Then $\{x_0\}$ is an isolated invariant set and*

$$\underset{i}{Con}(\{x_0\}, \pi_f) = \begin{cases} 0 & \text{for } i \neq k \\ (\mathbb{Q}, (-1)^l\, id) & \text{for } i = k, \end{cases}$$

*where (k,l) is the Morse index of $\{x_0\}$.*

# 6 The Conley index for multivalued discrete dynamical systems.

The only way to obtain some rigorous information from numerical computations is to perform rigorous error analysis. Such an analysis is possible (see []). As an outcome we obtain a multivalued map: the exact values are not known but a set enclosing the exact values is computer rigorously. This makes necessary extending the Conley index theory to multivalued maps. Such a theory for multivalued flows was proposed in [57]. The generalization in that case is quite natural. However, the discrete case is different, because it is not evident how to define an isolating neighbourhood in that case.

Let us recall that a *multivalued (mv) map* is a map $F : \mathbb{R}^p \to \mathcal{P}\mathbb{R}^q$. For $A \subset \mathbb{R}^p, B \subset \mathbb{R}^q$ the *image of* $A$, *weak preimage of* $B$ *and strong preimage of* $B$ are defined by

$$F(A) := \bigcup \{\, F(x) \mid x \in A \,\}$$

$$F^{*-1}(B) := \{\, x \in X \mid F(x) \cap B \neq \emptyset \,\}$$

$$F^{-1}(B) := \{\, x \in \operatorname{dom} F \mid F(x) \subset B \,\}$$

$F$ is *upper semicontinuous (usc)* if $F^{-1}(U)$ is open for any open $U \subset \mathbb{R}^q$.

In [31] an extension of the Conley index theory, based on the following definition of the isolating neighbourhood, was proposed.

**Definition 6.1** *Assume $F : X \to \mathcal{P}X$ is a multivalued map and $N \subset X$. The set $Inv(N, F)$ is defined as the set of $x \in N$ such that there exists a function $\sigma : Z \to N$ satisfying $\sigma(0) = x$ and $\sigma(n+1) \in F(\sigma(n))$. The set $N$ is said to be an isolating neighbourhood for $F$ if*

$$B_{diam_N F}(Inv\, N) \subset int\, N,$$

*where $diam_N\, F$ is the maximal diameter of the values of $F$ in $N$.*

The generalization of the notion of the index pair is then straightforward:

**Definition 6.2** Let $N$ be an isolating neighbourhood for $F$. A pair $P = (P_1, P_2)$ of compact subsets $P_2 \subset P_1 \subset N$ is called an *index pair* if the following conditions are satisfied:

$$F(P_i) \cap N \subset P_i,\ \ i = 1, 2;$$

$$F(P_1 \backslash P_2) \subset N;$$
$$\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2)$$

The definition of the index map does not essentially differ from the single valued case. One only needs to put some admissibility conditions to ensure that the multivalued map induces a map on cohomology level. This is all what is necessary to develop the Conley index theory for multivalued discrete dynamical systems.

Obviously not all multivalued maps may be treated by computer. To do this some finite way of representing these maps is necessary. For this end let us introduce the following definitions.

Let $\hat{\mathbb{R}} \subset \mathbb{R}$ be a given finite set. *Elementary representable sets (over* $\hat{\mathbb{R}}$*)* are the elements of

$$\mathcal{E} := \{A_1 \times A_2 \times \cdots \times A_d \mid$$
$$A_i = \{a\} \text{ or } A_i = (a, b) \text{ where } a, b \in \hat{\mathbb{R}},\ a << b\},$$

where $a << b$ means that $a$ is an immediate predecesor of $b$.

*Representable sets* are the elements of

$$\mathcal{R} := \{E_1 \cup E_2 \cup \ldots \cup E_k \mid E_i \in \mathcal{E}\}$$

A mv map $F : \mathbb{R}^p \to \mathcal{P}\mathbb{R}^q$ is *representable* if

$$\operatorname{dom} F := \{x \in \mathbb{R}^p \mid F(x) \neq \emptyset\} \in \mathcal{R},$$
$$\forall x \in \operatorname{dom} F \ F(x) \in \mathcal{R},$$
$$\forall E \in \mathcal{E} \ F_{|E} = \text{const},$$

If $U \subset \mathbb{R}^p$ and $f : U \to \mathbb{R}^q$ then $F : \mathbb{R}^p \to \mathcal{P}\mathbb{R}^q$ is a *representation of $f$* if $F$ is representable and $f$ is a *selector* of $F_{|U}$, i.e.

$$U \subset \operatorname{dom} F \text{ and } \forall x \in U \ f(x) \in F(x).$$

Assume that $\{\hat{\mathbb{R}}_n\}$ is a given, monotonically increasing sequence of finite subsets of $\mathbb{R}$ such that $\bigcup \{\hat{\mathbb{R}}_n\}$ is dense in $\mathbb{R}$. We have the following theorem.

**Theorem 6.3** *(see [31, 63]) If $f$ is L-Lipschitz continuous, then there exists a sequence $\{F_n\} : \mathbb{R}^p \to \mathcal{P}\mathbb{R}^q$ of usc multivalued representations of $f$ such that $F_n \to f$.*

The above theorem shows that, given a sufficiently powerful computer, any Lipschitz continuous map admits an arbitrarily close multivalued representation.

# 7 Inheritable properties.

Multivalued representations of single valued maps enable passing from the world of continua to the world of finite mathematics. To be able to go back one needs the concept of inheritance. Assume $\mathcal{A}$ is a collection of mv maps and $\varphi(F)$ is a property of such maps. We say that $\varphi$ is *inheritable* if for every single-valued selector $f$ of $F$ in $\mathcal{A}$

$$\varphi(F) \;\Rightarrow\; \varphi(f)$$

We say that an inheritable property $\varphi$ is *strongly inheritable* if for any single valued map $f \in \mathcal{A}$ such that $\varphi(f)$ and for any sequence $\{F_n\} \subset \mathcal{A}$ satisfying $F_n \to f$ we have $\varphi(F_n)$ for $n$ sufficiently large.

If $\alpha(F)$ is a term then we say that $\alpha$ is inheritable (strongly inheritable) if for any $x$ the property $\alpha(F) = x$ is inheritable (strongly inheritable).

**Example 7.1** Take $\mathcal{A} = \{F : \mathbb{R}^p \to \mathcal{P}\mathbb{R}\}$ and consider the following three properties

$$\alpha(F) \;\Leftrightarrow\; \exists x \in \mathbb{R}^p \; F(x) > 0$$

$$\beta(F) \;\Leftrightarrow\; \exists x \in \mathbb{R}^p \; F(x) \geq 0$$

$$\gamma(F) \;\Leftrightarrow\; \exists x \in \mathbb{R}^p \; F(x) \ni 0$$

The first property is strongly inheritable. The second property is inheritable but not strongly inheritable. The last property is not inheritable.

**Example 7.2** Assume $\mathcal{A} = \{F : \mathbb{R}^p \to \mathcal{P}\mathbb{R}^p\}$ and let $B$ be the unit ball in $\mathbb{R}^p$. Consider the following three properties

$$\mu(F) \;\Leftrightarrow\; F(B) \subset \operatorname{int} B$$

$$\nu(F) \;\Leftrightarrow\; F(B) \subset B$$

$$\kappa(F) \;\Leftrightarrow\; \exists x \in B \; x \in F(x)$$

Again the first property is strongly inheritable, the second property is inheritable but not strongly inheritable and the last property is not inheritable.

**Remark 7.3**

1. Multivalued representable maps behave like combinatorial objects; operations on them are algorithmizable.

2. Multivalued representable usc maps are conceptually close to ordinary continuous maps, so some theories carry over from single-valued continuous maps to multivalued usc maps. This allows for many natural inheritable properties.

3. Multivalued representations of single valued maps may be used to prove inheritable properties of single valued maps.

4. If a single valued map satisfies a strongly inheritable property then a sufficiently powerful computer will prove it.

5. Usually properties of real interest are non-inheritable

6. To get interesting applications we need theorems which convert inheritable properties to non-inheritable properties

An example of such a theorem, which converts property $\alpha$ in Example 7.1 to property $\gamma$, is the Darboux Theorem. Similarly, the theorem, which converts property $\mu$ in Example 7.2 to property $\kappa$ is the Brouwer Fixed Point Theorem.

The following theorem is essential in applications of the Conley index to rigorous numerics.

**Theorem 7.4** *(see [63]). The isolating neighborhood, the index pair, and the Conley index are strongly inheritable terms.*

Of course, in order to be able to compute the Conley index of a multivalued representation we need to know if there exists an index pair consisting of representable sets and how to construct it.

This is relatively easy under extra assumptions as in the following theorem.

**Theorem 7.5** *(see [63]) If $F : \mathbb{R}^d \to \mathcal{P}\mathbb{R}^d$ is usc representable and $N$ is a representable set such that*

$$B_{diam_N F}(F^{*-1}(N) \cap N \cap F(N)) \subset int\, N.$$

*then $N$ is an isolating neighborhood and $(N, N \backslash F^{-1}(int\, N))$ is an index pair in $N$ consisting of representable sets.*

A general answer to the question of the existence and construction of representable index pairs may be found in [64].

# 8 Chaos in the Lorenz equations

The world of representable multivalued maps and inheritable properties opens the way to rigorous computations of the Conley index for concrete dynamical systems. Since there are various qualitative descriptions of dynamics in terms of the Conley index, many new interesting results may be obtained this way. As an example let us announce a result concerning chaotic dynamics in the classical case of the Lorenz equations (see [53] and also [51, 52]). The result is based on a generalization of the Conley index theory proposed by Szymczak [92].

For a $k \times k$ matrix $A = (A_{ij})$ over $\mathbb{Z}_2$ put

$$\Sigma(A) := \{\alpha : \mathbb{Z} \to \{1, 2, \ldots, k\} \mid \forall i \in \mathbb{Z} A_{\alpha(i)\alpha(i+1)} = 1\}$$

$$\sigma : \Sigma(A) \ni \alpha \to (\mathbb{Z} \ni i \to \alpha(i+1) \in \{1, 2, \ldots k\} \in \Sigma(A).$$

**Theorem 8.1** *(see [53]). Consider the Lorenz equations*

$$\begin{aligned} \dot{x} &= s(y - x) \\ \dot{y} &= Rx - y - xz \\ \dot{z} &= xy - qz, \end{aligned} \tag{1}$$

*and the plane* $P := \{(x, y, z) \mid z = 27\}$*. For all parameter values in a sufficiently small neighborhood of* $(s, R, q) = (28, 10, 8/3)$ *there exists a Poincaré section* $N \subset P$ *such that the associated Poincaré map is Lipschitz and well defined. Furthermore, for*

$$A := \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

*there is a continuous surjection* $\rho : Inv(N, g) \to \Sigma(A)$ *such that*

$$\rho \circ g = \sigma \circ \rho.$$

*Moreover, for every* $\alpha \in \Sigma(A)$ *which is periodic there exists an* $x \in Inv(N, g)$ *on a periodic trajectory such that* $\rho(x) = \alpha$.

The main tool in the proof of the above theorem is a criterion of chaos (see [53]) based on the Conley index theory. In order to verify the assumptions of that theorem we choose a candidate for a representable isolating neighborhood $N$ for the Poincare map $f$ and compute a multivalued representation $F$ of $f$. If $N$ turns out to be an isolating neighborhood for $F$, we compute an index pair and find the Conley index for $F$. By inheritance, $f$ has the same Conley index in $N$. If the computed index satisfies the assumptions of the choas criterion, which indeed is the case, the theorem is proved. If not, the proof fails. In case $N$ fails to be an isolating neighborhood for $F$, one is free to find a better mv representation of $f$ and repreat the verification process.

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