Heteroclinic connections in the Kuramoto-Sivashinsky equation: a computer assisted proof

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Abstract. On the example of a finite dimensional approximation of the Kuramoto-Sivashinsky equation we show how topological methods may be successfully used in computer assisted proofs of the existence of heteroclinic connections in ordinary differential equations.

Key words: Conley index, isolating neighborhood, heteroclinic connection, computer assisted proof

1. Introduction

Consider the differential equation

$$x' = f(x), \tag{1}$$

where $x \in R^p$ and $f : R^p \to R^p$ is a continuous vector field. The rarely reachable ultimate goal of the theory of differential equations is to find all solutions of the given equation. For many applications it is sufficient to find just one particular solution satisfying some special conditions but, as far as analytic methods are concerned, the job is not easier. Thus the commonly used approach is to apply numerical methods to construct approximate solutions satisfying requested conditions. Unfortunately in such an approach, there is no way to tell what is the limit behavior of the trajectory, because the approximate solutions are constructed over a finite time interval.

Moreover, there are many situations, where finding just one or a few particular solutions does not provide any interesting information. This often happens when the vector field f is known only roughly. What really matters then is the understanding of the asymptotic behavior of solutions, if possible all solutions. In other words one would like to know what happens to a generic trajectory. Does it escape to infinity, limit in a stationary point? or a periodic trajectory? Or maybe

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it eventually exhibits periodic or chaotic behavior? The theory which studies differential equations from this point of view is know as the theory of dynamical systems or qualitative theory of differential equations. Of course the mentioned question are very ambitiuous and one usually starts with simpler problems like: does the equation have bounded solutions? bounded nontrivial solutions? What is the structure of the set of bounded solutions? Are among them periodic solutions? homoclinic or heteroclinic connections?, invariant tori?

Despite the enormous number of papers devoted to these problems the theory of dynamical systems is still far from being satisfactory, especially when concrete differential equations are concerned. And even if some methods work for some concrete equations, it is usually difficult to carry them over to other problems. Thus quite often the only available method is to experiment numerically with possibly many solutions, studied on long time intervals. Unfortunately, such an approach cannot be treated as reliable. Whatever is the length of the time interval and the number of studied trajectories, both, the length and the number are finite, which is always far from the global asymptotic behavior. Additionally, the computed trajectories are only approximate.

All this makes that the problems in dynamical systems constitute a natural field of research in rigorous numerics. However only recently some new techniques were developed, for which the amount of computations necessary is in the reach of present days computers (see [9, 5, 11]). Especially powerful seem to be the methods based on topological invariants like the Conley index [5] and the fixed point index [11].

In this paper we want to give an example of a topological approach based on the notion of an isolating neighborhood. Unlike [5], the computational part of the presented approach is easy to understand and should constitute a good introduction to topological methods in rigorous numerics of dynamical systems.

2. Isolating neighborhoods.

In the sequel we assume that for every $x \in R^p$ the equation (1) admits a unique solution $\varphi_x : R \to R^p$ satisfying $\varphi_x(0) = x$. This solution will be called the trajectory of x. One easily verifies that the function

$$\varphi : R^p \times R \ni (x,t) \to \varphi(x,t) := \varphi_x(t)$$

satisfies the following two properties

$$\varphi(x,0) = x$$

$$\varphi(\varphi(x,t),s) = \varphi(x,t+s).$$

Actually, a continuous function φ which satisfies the above two properties is a natural abstraction of the idea of the solution of an autonomous differential equation and is called a *dynamical system* or briefly a *flow*. The flow obtained by means of the differential equation (1) is called the flow generated by (1).

Let $N \subset R^p$ be a compact set. The invariant part of N with respect to φ is defined by

$$\mathrm{Inv}N := \mathrm{Inv}(N, \varphi) := \{\, w \in R^p \mid \forall t \in R\ \varphi(w, t) \in N \,\}.$$

The set N is an *isolating neighborhood* iff $\mathrm{Inv}N \subset \mathrm{int}N$.

At the first glance the notion of the isolating neighborhood doesn't look like something suitable for rigorous computation, because the definition of $\mathrm{Inv}N$ requires the understanding of the behavior of a trajectory over an infinite period of time. Fortunately there is an equivalent definition.

We will say that the trajectory of $x \in N$ leaves N if there exists a $t \in R$ such that $\varphi(x, t) \notin N$. We will say that the trajectory leaves N immediately if t may be chosen arbitrarily small. We will say that the trajectory of x leaves N in positive (negative) time if t may be chosen positive (negative). The reader will check easily that a compact set $N \subset R^p$ is an *isolating neighborhood* with respect to φ iff every point in the boundary of N leaves N.

Notice that if a trajectory of x leaves N then this fact may be confirmed rigorously in finite computation. Indeed, if x leaves N then $\varphi(x, t) \notin N$ for some $t \in R$ and since N is compact, there is a $\delta > 0$ such that $B(\varphi(x, t), \delta)$, the ball centered at $\varphi(x, t)$ of radius δ is disjoint from N. Thus in order to rigorously confirm that the trajectory leaves N, it is enough to rigorously compute $\varphi(x, t)$ with precision better than δ. The machinery neededed to confirm this task is well developed (see [2]). Moreover, since φ is continuous, the computation applies not only to x but also to a ball around x. If the vector field is Lipschitz then one can estimate the radius of that ball from below. Thus in order to check that a given set is an isolating neighborhood it is enough to cover the boundary of N with a finite number of such balls. If N is an isolating neighborhhood, then the computation may be performed in finite time and the only potential obstacle is the amount of necessary computations.

If rigor is disregarded then it is easy to perform numerical experiments and they indicate that the isolating neighborhood is a fairly common phenomenon. Unfortunately, it is usually very hard to prove it by means of purely analytic tools.

Observe that the trajectory of every point in the boundary of N falls into one of the following five categories

–an exit point ($\exists\epsilon > 0 \;\; t \in (0, \epsilon) \;\Rightarrow\; \varphi(x, t) \notin N$),

–an entrance point ($\exists\epsilon > 0 \;\; t \in (0, \epsilon) \;\Rightarrow\; \varphi(x, t) \in N$) ,

–an external tangency ($\exists\epsilon > 0 \;\; |t| \in (0, \epsilon) \;\Rightarrow\; \varphi(x, t) \notin N$)

–an internal tangency ($\exists\epsilon > 0 \;\; |t| \in (0, \epsilon) \;\Rightarrow\; \varphi(x, t) \in N$)

–a singular point, if none of the above is true.

Observe that singular ponts may be present even if the flow is very regular. It is worth to consider the example of a flow in the plane moving with constant non-zero speed along horizontal line where the compact set N has the form of a square with part of the top side replaced by a piece of the graph of ($x \to x^2 \sin(1/x)$. Nevertheless, if a point in the boundary of N is not an internal tangency then its trajectory leaves N immediately in positive, negative or both directions, even if the point is singular. Thus if N is not an isolating neighborhood then there must be a point of internal tangency somewhere in the boundary of N. Usually it is relatively easy to separate points in particular categories and the real effort in showing that a given set is an isolating neighborhood goes into proving that despite the presence of some internal tangencies, all trajectories originating in internal tangencies leave N eventually. It is usually hard, because it requires understanding the behavior of trajectories for a long period of time. Of course, the ideal situation is when there are no internal tangencies at all. Then the set N is obviously an isolating neighborhood, in this special case called an *isolating block*. Unfortunately, contrarily to isolating neighborhoods, isolating blocks are rare.

The main idea of the paper may be summarized in the following two points

1. We may draw conclusions concerning the global dynamics of a differential equation from the fact that certain sets are isolating neighborhoods.

2. The fact that a given set is an isolating neighborhood, though often difficult to show by analytic means, may be proved easily via rigorous numerics, the amount of necessary computations being the only potential barrier.

3. The Kuramoto-Sivashinsky equation.

We want to illustrate this idea on the example of the following differential equation.

$$\begin{cases} \dot{x_0} = x_1 \\ \dot{x_1} = x_2 \\ \dot{x_2} = -\lambda x_1 - \frac{x_0^2}{2} + 1. \end{cases} \tag{2}$$

One easily verifies that the equation has exactly two stationary points: $x^- := (-\sqrt{2}, 0, 0), x^+ := (\sqrt{2}, 0, 0)$. An interesting question is if the equation admits a solution $x(t)$ such that

$$\lim_{t \to -\infty} x(t) = x^-, \quad \lim_{t \to \infty} x(t) = x^+. \tag{3}$$

Such a solution is called a heteroclinic connection. A numerical evidence of the existence of such a solution in the equation is shown in Fig. 1.

In case of $\lambda = 1$ equation (2) is the ODE derived from the Kuramoto-Sivashinsky PDE and studied numerically in [3, 4]. It is shown in [10] that for $\lambda = 1$ the equation (2) admits heteroclinic connections. In case of $\lambda = 0$ the equation becomes a variant of an equation studied by C. Conley, who shows that it also admits a heteroclinic connection. It turns out that combining the technique of Conley and some ideas of Troy with a computer assisted proof that certain sets are isolating neighborhoods one can show that (2) admits a heteroclinic connection for every $\lambda \in [0, 1]$. In other words the following generalization of the results of Troy and Conley may be proven.

*Theorem 3.1.*For every $\lambda \in [0, 1]$ there exists a solution $x_\lambda(t)$ of the equation (2) such that (3) is satisfied.

The first step in the proof of the above theorem consists in the change of variables $y_0 := \delta^3 x_0$, $y_1 := \delta^4 x_1$, $y_2 := \delta^5 x_2$, $\tau := t/\delta$ which results in the following system conjugated to (2) for $\delta \in R \setminus \{0\}$

$$\begin{cases} \dot{y_0} = y_1 \\ \dot{y_1} = y_2 \\ \dot{y_2} = -\delta^2 \lambda y_1 - \frac{y_0^2}{2} + \delta^6. \end{cases} \tag{4}$$

where differentiation is now with respect to the τ variable.

For $a, b, c, \epsilon > 0$ put

$$N_{abc} := [-a, a] \times [-b, b] \times [-c, c]$$

and

$$N_{abc}^{\epsilon} := N_{abc} \setminus \{\, (y_0, y_1, y_2) \mid y_1 > 0, |y_0| < \epsilon \,\}.$$

The proof of Theorem 3.1 requires the following result.

*Theorem 3.2.*There exist numbers $a_0, b_0, c_0, \delta_0, \epsilon_0 > 0$ such that

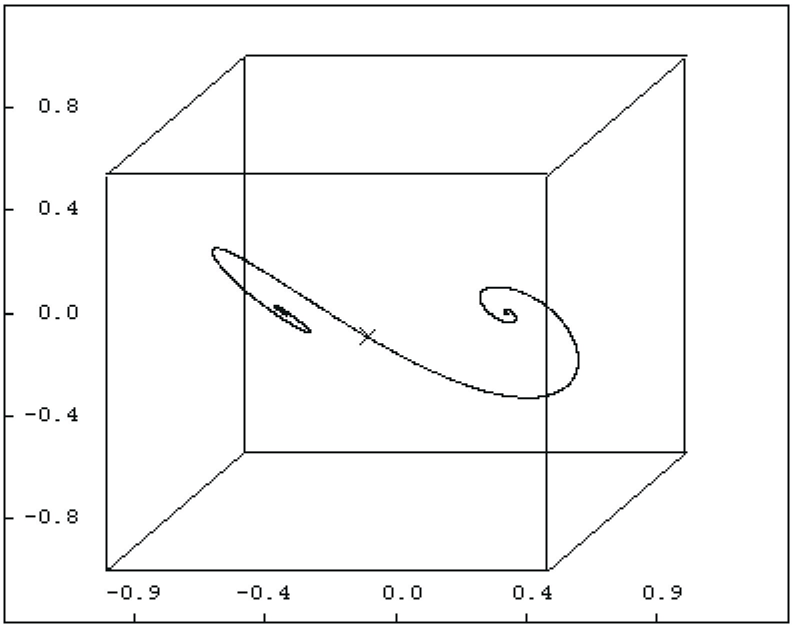

Figure 1. A numerical indication of the existence of a heteroclinic connection in the Kuramoto-Sivashinsky equation. In this and in the next picture the scaled framing lies in the plane $\{ y \in R^3 \mid y_1 = 0 \}$, the parameters are $\lambda = 1$ and $\delta = 0.75$, and the constants used to define the cube N_{abc} are $a = 1.04, b = 1.05, c = 1.0$.

(i)for $\lambda = 0$ and $\delta \in [0, \delta_0]$ the set $N_{a_0 b_0 c_0}$ is an isolating neighborhood with respect to the flow generated by the equation (4)

(ii)for $\lambda = 0$ and $\delta = \delta_0$ $\mathrm{Inv} N^{\epsilon}_{a_0 b_0 c_0} = \mathrm{Inv} N_{a_0 b_0 c_0}$.

(iii)for every $\lambda \in [0, 1]$, $\epsilon \in (0, \epsilon_0)$ and for $\delta = \delta_0$ the set $N^{\epsilon}_{a_0 b_0 c_0}$ is an isolating neighborhood with respect to (4)

In the sequel we will show how establishing the above three facts concerning isolation in the Kuramoto-Sivashinsky equation may be used to obtain Theorem 3.1 via topological methods. However first we want to sketch the proof of Theorem 3.2. The details are too lengthy for this short communication and will be presented in [12]. The first observation is that if the boundary of N is a smooth manifold and the vector field f at a certain point of the boundary is not tangent to the boundary then the trajectory of that point must leave the set N immediately either in positive or negative direction. Thus if N fails to be an isolating neighborhood then this is because of points at the boundary, where the vector field is tangent to the boundary and the tangency is internal. (If the tangency is external then the trajectory leaves N both in positive and negative direction). If the boundary of N is a manifold with corners, which happens in case of N_{abc} and N^{ϵ}_{abc}, the situation is technically slightly more complicated but conceptually similar.

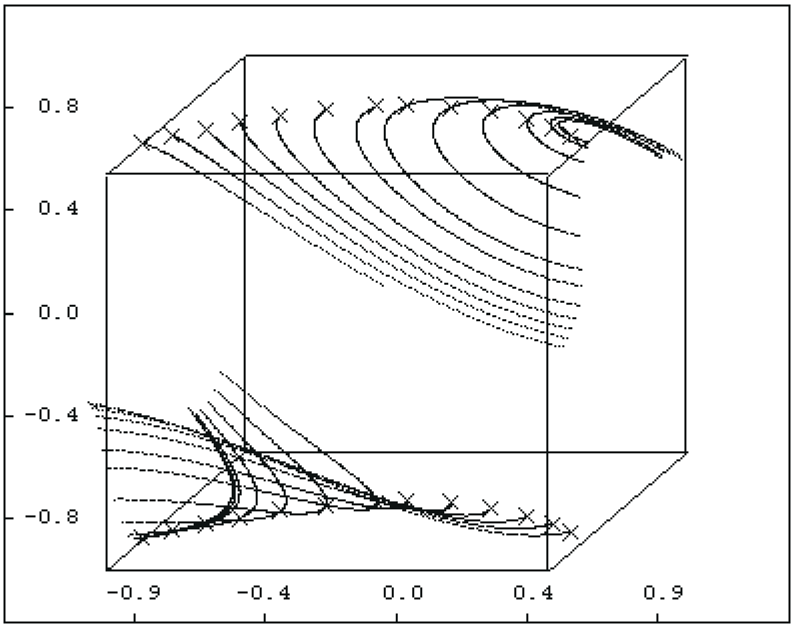

Figure 2. Bottom and top internal tangencies in the set N_{abc}. Trajectories originating from the top left and bottom right part of the parabolas are shown in one direction only since in the other direction the trajectories do not leave N_{abc} in reasonable time.

It is now a simple calculation to check that for the set $N_{a_0b_0c_0}$ all points of internal tangency of the flow generated by (4) are contained in one of the following three sets

$$\{\, y \in R^3 \mid y_0 = \pm a_0, y_1 = 0 \,\}, \;\; \{\, y \in R^3 \mid y_1 = \pm b_0, y_2 = 0 \,\}$$

$$\{\, y \in R^3 \mid y_2 = \pm c_0, -\delta^2 \lambda y_1 - \tfrac{y_0^2}{2} + \delta^6 = 0 \,\}.$$

The two parabolas with points of internal tangencies at the bottom and at the top of $N_{a_0b_0c_0}$ and the outcoming trajectories are shown in Figure (2).

Thus what remains to be done is to cover the above set by a certain amount of small balls (intervals) and integrate the equation from these balls forward and backward until the value is disjont from the set N. For details how to perform this task successfully in case of the Kuramoto-Sivashinsky equation the reader is refered to [12].

4. The Conley index.

We go now to the proof of Theorem 3.2. The proof follows the ideas of Conley [1] and is based on a topological invariant called after him the Conley index. In its simplest setting, it assigns to every isolating neighborhood N with respect to a flow φ a vector space denoted by $C(N, \varphi)$. Although the definition of the index is rather complicated, in order to successfully use it the knowledge of a few of its basic properties is sufficient for most applications.

The fundamental properties of the Conley index are

(i)**Wazewski property**: If $C(N, \varphi) \neq 0$ then $\mathrm{Inv}(N, \varphi) \neq 0$, i.e. there is a trajectory entirely contained in N.

(ii)**Stability property**: If N is an isolating neighborhood with respect to a flow φ then it is an isolating neighborhood for all nearby flows (in compact-open topology) and the Conley index for the nearby flows does not change.

(iii)**Homotopy property**: Assume φ_s, for $s \in [0,1]$, is a family of flows continuously depending on s. If $N \subset R^p$ is an isolating neighborhood with respect to every φ_s then $C(N, \varphi_s) = \mathrm{const}$.

(iv)**Additivity property**: If N, N_1, N_2 are three isolating neighborhoods such that

$$\mathrm{Inv} N = \mathrm{Inv} N_1 \cup \mathrm{Inv} N_2 \text{ and } \mathrm{Inv} N_1 \cap \mathrm{Inv} N_2 = \emptyset$$

then

$$C(N, \varphi) = C(N_1, \varphi) \oplus C(N_2, \varphi).$$

(v)**Normalization property**: Assume φ is given by the differential equation (1) and x^* is a hyperbolic stationary point of φ (i.e. $Df(x^*)$ has no purely imaginary eigenvalues). Then every sufficiently small compact neighborhood N of $\{x^*\}$ is an isolating neighborhood and $C(N, \varphi)$ is a non-zero vector space.

(vi)**Excision property**: If N_1 and N_2 are two isolating neighborhoods such that $\mathrm{Inv} N_1 = \mathrm{Inv} N_2$ then $C(N_1, \varphi) = C(N_2, \varphi)$.

As an excercise the reader may show that the excision property follows easily form the Wazewski property and the additivity property and the homotopy property from the stability property.

We are now ready to present the proof of Theorem 3.1.

Proof Theorem 3.1 Let $a_0, b_0, c_0, \delta_0, \epsilon_0$ be chosen so that the conclusion of Theorem 3.2 is satisfied. Let $\varphi_{\lambda,\delta}$ denote the flow generated by (4). First observe that $N_{a_0b_0c_0}$ is an isolating neighborhood with respect to $\varphi_{0,0}$, because the origin is the only bounded solution for this flow. Moreover, there are flows arbitrarily close to $\varphi_{0,0}$ which have no bounded solutions at all. It follows from the Wazewski property that $C(N_{a_0b_0c_0}, \varphi_{0,0}) = 0$. By Theorem 3.2,(i) and the homotopy property $C(N_{a_0b_0c_0}, \varphi_{0,\delta_0}) = 0$. Put $\epsilon := \epsilon_0/2$. By Theorem 3.2,(ii) and the excision property

$$C(N^{\epsilon}_{a_0b_0c_0}, \varphi_{0,\delta_0}) = C(N_{a_0b_0c_0}, \varphi_{0,\delta_0}) = 0$$

Finally by Theorem 3.2,(iii) and the continuation property again we get

$$C(N^{\epsilon}_{a_0b_0c_0}, \varphi_{\lambda,\delta_0}) = 0$$

for any $\lambda \in [0, 1]$.

An elementary calculation shows that both stationary points of $\varphi_{\lambda,\delta_0}$ are hyperbolic. The normalization property of the Conley index lets us choose compact neighborhoods N_1 and N_2 of these points which are isolating and small enough to be contained in $N^{\epsilon}_{a_0b_0c_0}$. We are able to show now that there is a point $y^{\#} \in R^3$ which has a non-stationary trajectory fully contained in $N := N^{\epsilon}_{a_0b_0c_0}$. Indeed, if this is not true, then the only trajectories fully contained in N are the two stationary points, and consequently $\mathrm{Inv}N = \mathrm{Inv}N_1 \cup \mathrm{Inv}N_2$. Thus, by the additivity property

$$C(N^{\epsilon}_{a_0b_0c_0}, \varphi_{\lambda,\delta_0}) = C(N_1, \varphi_{\lambda,\delta_0}) \oplus C(N_2, \varphi_{\lambda,\delta_0}) \neq 0,$$

a contradiction.

It remains to show that the non-stationary trajectory is indeed a heteroclinic connection. This is achieved via the following variant of a functional considered in [10].

$$H(y_0, y_1, y_2) := y_2^2/2 + y_1(\lambda\delta_0^2 y_1/2 - \delta_0^6 + y_0^2/2)$$

Put $V^+ := \{\, y \in R^3 \mid y_0 \geq 0 \,\}$ and $V^- := \{\, y \in R^3 \mid y_0 \leq 0 \,\}$. One easily verifies that the derivative of H along the trajectories of $\varphi_{\lambda,\delta_0}$ is non-negative in V^+ and non-positive in V^-. This means that H increases along the non-stationary trajectories of $\varphi_{\lambda,\delta_0}$ in V^+ and decreases in V^-. In particular, if a trajectory is relatively compact and entirely contained in V^- or entirely contained in V^+, then the trajectory is stationary (an alternative would be a heteroclinic connection joining two stationary points in V^- or two stationary points in V^+ but there is exactly one stationary point in V^- and exactly one stationary point in V^+. It follows that the trajectory of $y^{\#}$ is neither entirely contained in V^- nor in V^+. Thus it intersects somewhere the plane

$\{ y \in R^3 \mid y_0 = 0 \}$. However, since the trajectory also lies in $N^{\epsilon}_{a_0 b_0 c_0}$, there is only one such intersection point. (Note that the trajectories fully contained in $N^{\epsilon}_{a_0 b_0 c_0}$ can go only from V^+ to V^-; the unique way to go in the other direction is via the set $\{ y \in R^3 \mid y_0 = 0, y_1 \geq 0 \}$ but this set is disjoint from $N^{\epsilon}_{a_0 b_0 c_0}$.) Without loss of generality we may assume that the intersection point is exactly $y^\#$. Then the negative trajectory of $y^\#$ is contained in V^+ and the positive in V^-. Using the monotonicity properties of H one easily concludes that the trajectory of $y^\#$ is the required heteroclinic connection. □

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