Inheritable Properties and Computer Assisted Proofs in Dynamics

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0 Introduction.

Computer assisted proofs are becoming an important factor of present day mathematics. Since there exist short theorems having arbitrarily long proofs (a consequence of Gödel's incompletness theorems, see [4, 6, 23]) and the power of computers is growing exponentially, this shoudbe expected.

The general scheme for constructing computer assisted proofs is actually very simple and is a direct consequence of the fact that any computer may deal only with finite sets.

A proof recipe
1. Take a suitable theory which reduces the problem to a question concerning a given finite set.
2. Apply an appropriate algorithm to answer the question.

If the first step of the scheme is performed succesfully then the recipe is guaranteed to work, at least theoretically, because the case analysis algorithm may be always applied. In practice, the availability of a fast algorithm is often crucial. The scheme also shows that computers will never replace mathematitians, since both developing the reduction in Step 1 and constructing the algorithm in Step 2 are jobs for a mathematician. Actually, the job is often quite challenging as in the case of the proof of the four color hypothesis by Appel and Haken [1, 2], probably the most famous computer assisted proof.

Computer assisted proofs in discrete mathematics and algebra are relatively common. There are theories in which one can rigorously prove or refute any conjectured statement in a given class by reducing the problem to a question which may be answered by running an appropriate algorithm. An example is the algorithmic proof theory of hypergeometric identities (see [27]).

[1] Research supported by KBN, grant 0449/P3/94/06

The number of computer assisted proofs in analysis is relatively small, because the mathematics of continua is far apart from discrete mathematics. Nevertheless even here we observe progress. One way of concretizing the above proof recipe for applications in analysis follows.

A proof recipe in analysis
1. Reduce the problem to a strong inequality among two numbers.
2. Use interval arithmetic to check if the inequality is true.

Exactly this scheme was applied in the proof of Feigenbaum's conjectures by Lanford (see [12, 13]), which is one of the first computer assisted proofs in analysis. The proof consist in reducing the problem to the Banach contraction principle and applying computer to show that the considered map is actually a contraction, i.e. a Lipschitz map with Lipschitz constant less than one. Similar in spirit are computer assisted proofs in [5, 11].

Two other interesting computer assisted proofs by Neumaier and Rage [21] and by Rage, Neumaier and Schlier [22] concern the existence of chaos in the standard map and in a Hamiltonian system govering a molecular model. In both cases the proof consists in reducing the problem to Smale's criterion for chaos, i.e. verifying the existence of a transversal homoclinic point. Transversality means a strong inequality among two angles.

One can concretize further the previous recipe.

1. Reduce the problem to the existence of a zero of a real analytic function f.
2. Using interval arithmetic find $a, b \in \mathbf{R}$ such that $f(a) < 0, f(b) > 0$.
3. Apply the Darboux property.

This recipe was used in the proof of the existence of parameter values of the Lorenz equations for which there exists a trajectory homoclinic to the origin. The first proof of this fact was given by Hastings and Troy [9]. They apply topological shooting, thus reduce the problem again to a strong inequality among two numbers. Theoretically it is possible to check the inequality by hand but the very nature of the proof calls for a computer. Actually Hassard and

Zhang [7], following the ideas of Hastings and Troy but employing a computer, obtained much narrower bounds for the parameters admitting the homoclinic trajectory. Similar methods where used to show complicated behaviour of solutions of the Lorenz equations in [8].

The aim of this note is to show that computer assisted proofs in analysis based on different principles than just verifying strong inequalities are possible. The recipe we propose is

1. Find a theory which spans the gap between the mathematics of continua and the mathematics of finite sets.
2. Translate your problem to a corresponding problem in finite mathematics.
3. Solve the finite problem.
4. Translate the solution back to the original setting.

Although we believe the above scheme may be concretized in many ways and in many areas of continuous mathematics, in the sequel we restrict our attention to some problems in dynamics. The theory which we propose as a bridge between the mathematics of continua and the mathematics of finite sets is the theory of multivalued representable maps.

1 Representable multivalued maps.

Assume $f : \mathbf{R}^p \multimap \mathbf{R}^q$ is an L-Lipschitz continuous partial map defined on a compact subset of $\mathbf{R}^p$. Often the interval arithmetic combined with the classical numerical analysis allows us to find rigorously enclosures of values of f over a finite subset in its domain. This way we obtain an object, a multivalued map defined over a finite set, which belongs to finite mathematics but at the same time is very far from the original map. We need something which is still finite but may share some features of objects in the mathematics of continua, such as the continuity property. For this end we cover the domain of f with a finite family of sets, for each set we compute an enclosure of the value of f at a point selected in this set and then, using the Lipschitz constant, we extend this enclosure so that it applies to all arguments in this set. This way we obtain a multivalued map which is still finite in the sense that it may be easily coded with finite amount of information, but still some concepts of continuity make sense for this map.

To be more concrete let us make the following definitions.

Let $\hat{\mathbf{R}} \subset \mathbf{R}$ be a given finite set. *Elementary representable sets (over $\hat{\boldsymbol{R}}$)* are the elements of

$$\mathcal{E} := \{A_1 \times A_2 \times \cdots \times A_d \mid$$
$$A_i = \{a\} \text{ or } A_i = (a,b) \text{ where } a,b \in \hat{\mathbf{R}},\ a << b\},$$

where $a << b$ means that a is an immediate predecesor of b.

Representable sets are the elements of

$$\mathcal{R} := \{E_1 \cup E_2 \cup \ldots \cup E_k \mid E_i \in \mathcal{E}\}$$

A *multivalued (mv) map* is a map $F : \mathbf{R}^p \to \mathcal{P}(\mathbf{R}^q)$. For $A \subset \mathbf{R}^p, B \subset \mathbf{R}^q$ the *image of A, weak preimage of B and strong preimage of B* are defined by

$$F(A) := \bigcup \{\, F(x) \mid x \in A \,\}$$
$$F^{*-1}(B) := \{\, x \in X \mid F(x) \cap B \neq \emptyset \,\}$$
$$F^{-1}(B) := \{\, x \in \operatorname{dom} F \mid F(x) \subset B \,\}$$

F is *upper semicontinuous (usc)* if $F^{-1}(U)$ is open for any open $U \subset \mathbf{R}^q$.

A mv map $F : \mathbf{R}^p \to \mathcal{P}(\mathbf{R}^q)$ is *representable* if

$$\operatorname{dom} F := \{x \in \mathbf{R}^p \mid F(x) \neq \emptyset\} \in \mathcal{R},$$
$$\forall x \in \operatorname{dom} F\ F(x) \in \mathcal{R},$$
$$\forall E \in \mathcal{E}\ F_{|E} = \text{const},$$

If $U \subset \mathbf{R}^p$ and $f : U \to \mathbf{R}^q$ then $F : \mathbf{R}^p \to \mathcal{P}(\mathbf{R}^q)$ is a *representation of f* if F is representable and f is a *selector* of $F_{|U}$, i.e.

$$U \subset \operatorname{dom} F \text{ and } \forall x \in U\ f(x) \in F(x).$$

Assume that $\{\hat{\mathbf{R}}_n\}$ is a given, monotonically increasing sequence of representations of $\mathbf{R}$ such that $\bigcup \{\hat{\mathbf{R}}_n\}$ is dense in $\mathbf{R}$. Formalizing the construction in the begining of this section yields the following theorem.

Theorem 1.1 *(see [19]) If f is L-Lipschitz continuous, then there exists a sequence $\{F_n\} : \boldsymbol{R}^p \to \mathcal{P}(\boldsymbol{R}^q)$ of usc multivalued representations of f such that $F_n \to f$.*

2 Inheritable properties.

Multivalued representations of single valued maps enable passing from the world of continua to the world of finite mathematics. To be able to go back one needs the concept of inheritance. Assume $\mathcal{A}$ is a collection of mv maps and $\varphi(F)$ is a property of such maps. We say that φ is *inheritable* if for every single-valued selector f of F in $\mathcal{A}$

$$\varphi(F) \;\Rightarrow\; \varphi(f)$$

We say that an inheritable property φ is *strongly inheritable* if for any single valued map $f \in \mathcal{A}$ such that $\varphi(f)$ and for any sequence $\{F_n\} \subset \mathcal{A}$ satisfying $F_n \;\to f$ we have $\varphi(F_n)$ for n sufficiently large.

If $\alpha(F)$ is a term then we say that α is inheritable (strongly inheritable) if for any x the property $\alpha(F) = x$ is inheritable (strongly inheritable).

Example 2.1 Take $\mathcal{A} = \{F : \mathbf{R}^p \to \mathcal{P}(\mathbf{R})\}$ and consider the following three properties

$$\alpha(F) \;\Leftrightarrow\; \exists x \in \mathbf{R}^p\; F(x) > 0$$

$$\beta(F) \;\Leftrightarrow\; \exists x \in \mathbf{R}^p\; F(x) \geq 0$$

$$\gamma(F) \;\Leftrightarrow\; \exists x \in \mathbf{R}^p\; F(x) \ni 0$$

The first property is strongly inheritable. The second property is inheritable but not strongly inheritable. The last property is not inheritable.

Example 2.2 Assume $\mathcal{A} = \{F : \mathbf{R}^p \to \mathcal{P}(\mathbf{R}^p)\}$ and let B be the unit ball in $\mathbf{R}^p$. Consider the following three properties

$$\mu(F) \;\Leftrightarrow\; F(B) \subset \operatorname{int} B$$

$$\nu(F) \;\Leftrightarrow\; F(B) \subset B$$

$$\kappa(F) \;\Leftrightarrow\; \exists x \in B\; x \in F(x)$$

Again the first property is strongly inheritable, the second property is inheritable but not strongly inheritable and the last property is not inheritable.

Remark 2.3

1. Multivalued representable maps behave like combinatorial objects; operations on them are algorithmizable.

2. Multivalued representable usc maps are conceptually close to ordinary continuous maps, so some theories carry over from single-valued continuous maps to multivalued usc maps. This allows for many natural inheritable properties.

3. Multivalued representations of single valued maps may be used to prove inheritable properties of single valued maps.

4. If a single valued map satisfies a strongly inheritable property then a sufficiently powerful computer will prove it.

5. Usually properties of real interest are non-inheritable

6. To get interesting applications we need theorems which convert inheritable properties to non-inheritable properties

An example of such a theorem, which converts property α in Example 2.1 to property γ, is the Darboux Theorem. Similarly, the theorem, which converts property μ in Example 2.2 to property κ is the Brouwer Fixed Point Theorem.

3 Isolating neighborhoods and the Conley index.

The range of applications of Darboux Theorem and Brouwer Fixed Point Theorem in dynamics is rather limited. We want to present briefly a theory which has much broader applications and fits into the approach presented.

Let $f : \mathbf{R}^d \to \mathbf{R}^d$ be a fixed homeomorphism generating a *discrete dynamical system*

$$f : X \times \mathbf{Z} \ni (x, n) \to f(x, n) := f^n(x) \in X.$$

Let $N \subset X$ be a compact. The *invariant set* of N is

$$\operatorname{Inv} N := \operatorname{Inv}(N, f) := \{x \in N \mid \forall n \in \mathbf{Z}\ f^n(x) \in N\}$$

The set N is called an *isolating neighborhood* if

$$\operatorname{Inv} N \subset \operatorname{int} N.$$

If N is an isolating neighborhood then the set $S := \operatorname{Inv} N$ is called an *isolated invariant set.*

The pair $P = (P_1, P_2)$ of compact subsets of N is an *index pair* of S in N if

$$x \in P_i,\ f(x) \in N \ \Rightarrow\ f(x) \in P_i, \quad i = 1, 2$$
$$x \in P_1,\ f(x) \notin N \ \Rightarrow\ x \in P_2$$
$$\operatorname{Inv} N \subset \operatorname{int}(P_1 \backslash P_2).$$

Theorem 3.1 *(see [18]) Every isolating neighborhood admits at least one index pair P.*

It can be easily derived from the definition of the index pair that f induces a map of pairs

$$f_P : (P_1, P_2) \ni x \to f(x) \in (P_1 \cup f(P_2), P_2 \cup f(P_2))$$

and the inclusion

$$i_P : (P_1, P_2) \ni x \to x \in (P_1 \cup (P_2), P_2 \cup f(P_2))$$

induces an isomorphism in Alexander-Spanier cohomology. The endomorphism

$$H^*(f_P) \circ H^*(i_P)^{-1} \text{ of } H^*(P),$$

where H^* denotes the Alexander-Spanier cohomology, is called the index map associated with the index pair P and denoted by I_P.

The *cohomological Conley index* of an isolating neighborhood N under f takes the form

$$\operatorname{Con}^*(\operatorname{Inv}(N, f)) = \Big(CH^*(\operatorname{Inv}(N, f)), \chi^*(\operatorname{Inv}(N, f))\Big),$$

where $CH^*(\operatorname{Inv}(N, f))$ is the graded module obtained by quotienting $H^*(P_1/P_2)$ by the generalized kernel of I_P (the union of the kernels of all iterates of I_P) and $\chi^*(\operatorname{Inv}(N, f))$ is the induced graded module automorphism on $CH^*(\operatorname{Inv}(N, f))$.

The Conley index, originally developed by C. Conley for flows (see [3]), turned out to be a very useful tool in proving the existence of various qualitative phenomena of dynamical system like the existence of stationary, bounded, periodic, heteroclinic or homoclinic trajectories. Recently it turned out that it may also be used to prove the existence of semi-conjugacies. Here is an example of a recent theorem which characterizes the existence of a semiconjugacy to chaotic dynamics in terms of Conley indexes.

Theorem 3.2 *(see [14]) Assume that $N = N_0 \cup N_1$ is an isolating neighborhood under f where N_0 and N_1 are disjoint compact sets and for $k = 0, 1$*

$$Con^n(N_k) = \begin{cases} (Q, Id) & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $N_{kl} := (N_k \cap f(N_k)) \cup (N_k \cap f(N_l)) \cup (N_l \cap f(N_l))$ for $k, l \in \{0, 1\}, k \neq l$, are isolating neighborhoods and if additionally $\chi^*(N_{kl})$ is not conjugate to identity, then there exists a $d \in N$ and a continuous surjection $\rho : Inv(N, f) \to \Sigma_2$ such that

$$\rho \circ f^d = \sigma \circ \rho,$$

where $\sigma : \Sigma_2 \to \Sigma_2$ is the full shift dynamics on two symbols.

In the sequel we want to show how the Conley index theory may be extended to multivalued dynamics and applied to computer assisted proofs.

4 Multivalued dynamics.

An usc compact-valued map $F : \mathbf{R}^d \to \mathcal{P}(\mathbf{R}^d)$ generates a *multivalued discrete semidynamical system (mvdss)*

$$F : X \times \mathbf{Z}^+ \ni (x, n) \to F(x, n) := F^n(x) \in X.$$

Define the *invariant set* of N by

$$\mathrm{Inv}(N, F) := \{x \in N \mid \exists \sigma : \mathbf{Z} \to N \text{s.t. } \sigma(0) = x \text{ and } \sigma(n+1) \in F(\sigma(n))\}.$$

The set N is an *isolating neighborhood* for F if

$$B_{\mathrm{diam}_N F}(\mathrm{Inv}\, N) \subset \mathrm{int}\, N.$$

A pair $P = (P_1, P_2)$ of compact subsets $P_2 \subset P_1 \subset N$ is an *index pair* if

$$F(P_i) \cap N \subset P_i, \ i = 1, 2;$$

$$F(P_1 \backslash P_2) \subset N;$$

$$\mathrm{Inv}\, N \subset \mathrm{int}(P_1 \backslash P_2)$$

Using the above two definitions one can carry over the Conley index theory to multivalued dynamics (see [10] for details).

The following theorem is essential in applications to computer assisted proofs.

Theorem 4.1 *(see [19]). The isolating neighborhood, the index pair, and the Conley index are strongly inheritable terms.*

Of course, in order to be able to compute the Conley index of a multivalued representation we need to know if there exists an index pair consisting of representable sets and how to construct it.

This is relatively easy under extra assumptions as in the following theorem.

Theorem 4.2 *(see [19]) If $F : \boldsymbol{R}^d \to \mathcal{P}(\boldsymbol{R}^d)$ is usc representable and N is a representable set such that*

$$B_{diam_N F}(F^{*-1}(N) \cap N \cap F(N)) \subset int N.$$

then N is an isolating neighborhood and $(N, N \setminus F^{-1}(int N))$ is an index pair in N consisting of representable sets.

A general answer to the question of the existence and construction of representable index pairs may be found in [25, 20].

5 Chaos in the Lorenz equations

Theorem 3.2 was applied in a computer assisted proof to obtain chaotic dynamics in the Lorenz equations for a certain, non-classical choice of parameter values (see [15, 16]). We wolud like to announce here a much stronger result concerning the classical case of the Lorenz equations. The result is based on a generalization of the Conley index theory proposed by Szymczak [24].

For a $k \times k$ matrix $A = (A_{ij})$ over $\mathbf{Z}_2$ put

$$\Sigma(A) := \{\alpha : \mathbf{Z} \to \{1, 2, \ldots, k\} \mid \forall i \in \mathbf{Z} A_{\alpha(i)\alpha(i+1)} = 1\}$$

$$\sigma : \Sigma(A) \ni \alpha \to (\mathbf{Z} \ni i \to \alpha(i+1) \in \{1, 2, \ldots k\} \in \Sigma(A).$$

Theorem 5.1 *(see [17]). Consider the Lorenz equations*

$$\begin{aligned} \dot{x} &= s(y - x) \\ \dot{y} &= Rx - y - xz \\ \dot{z} &= xy - qz, \end{aligned} \tag{1}$$

and the plane $P := \{(x, y, z) \mid z = 27\}$. For all parameter values in a sufficiently small neighborhood of $(s, R, q) = (28, 10, 8/3)$ there exists a Poincaré section $N \subset P$ such that the associated Poincaré map is Lipschitz and well defined. Furthermore, for

$$A := \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

there is a continuous surjection $\rho : Inv(N, g) \to \Sigma(A)$ such that

$$\rho \circ g = \sigma \circ \rho.$$

Moreover, for every $\alpha \in \Sigma(A)$ which is periodic there exists an $x \in Inv(N, g)$ on a periodic trajectory such that $\rho(x) = \alpha$.

The proof of this theorem is based on the following scheme.

1. Choose a candidate for a representable isolating neighborhood N for the Poincare map f.
2. Compute a mv representation F of f.
3. Check if N is an isolating neighborhood for F.
4. If no, find a better mv representation of f and go back to 3.
5. If yes, compute the index pair and find the Conley index.
6. By inheritance, f has the same Conley index in N.
7. If the computed index satisfies the assumption of the choas criterion, proclaim SUCCESS, otherwise proclaim FAILURE.

The computations were performed on SUN SPARC work stations and IBM PC's. The CPU time depends on parameters chosen.

CPU time of PENTIUM 90MHz	
R=54,s=45,q=10	1.25h
R=28,s=10,q=8/3	3.5h
R=60,s=10,q=8/3	14h

6 The Kuramoto-Sivashinsky equation

The following PDE was derived by Kuramoto *et al.* (1976) as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky (1977) as a model for plane flame propagation.

$$u_t + \nabla^4 u + \nabla^2 u + \frac{1}{2}|\nabla u|^2 = 0.$$

We search for solutions of the form $u(x,t) = c^2 t + v(x)$ which gives

$$y''' + y' = c^2 - \frac{y^2}{2},$$

where $y = v'$.

The following theorem was obtained by Troy [26] in a calculator assisted proof based on the shooting technique.

Theorem 6.1 *There are at least two odd periodic solutions and at least two heteroclinic solutions of*

$$\begin{cases} y''' + y' = 1 - \frac{y^2}{2} \\ y(0) = y''(0) = 0 \end{cases} \tag{2}$$

In the same paper Troy proposed the following conjecture.

Conjecture 6.2 *Equation (2) has infinitely many periodic solutions.*

Numerical experiments indicate that there is horseshoe dynamics embedded in the Kuramoto-Sivashinsky equation. This justifies the following conjecture.

Conjecture 6.3 *Equation (2) admits a Poincaré section and an invariant set for the associated Poincaré map which is semi-conjugated to the shift dynamics on two symbols so that each periodic sequence is represented by a periodic point.*

Theoretically one could try to prove the above conjecture using the methods presented in this paper. The problem is that the computational complexity in this case is very high. Thus faster computers or better algorithms are needed.

The presence of y' in (2) is essential for the existence of periodic orbits, as the following theorem shows

Theorem 6.4 *Function* $V(y) := -\frac{y^3}{6} + y - y'y''$ *decreases along non-stationary trajectories of*

$$y''' = 1 - \frac{y^2}{2} \tag{3}$$

In particular equation (3) has no periodic solutions.

This makes the following conjecture interesting (Its computer assisted proof is in progress).

Conjecture 6.5 *Consider the family of equations*

$$\begin{cases} y''' + \alpha y' = 1 - \frac{y^2}{2} \\ y(0) = y''(0) = 0 \end{cases} \tag{4}$$

For all $\alpha \in [0,1]$ *there exists a solution to (4) satisfying* $y(+\infty) = -\sqrt{2},\ y(-\infty) = \sqrt{2}$. *There exists an* $\alpha \in [0,1]$ *such that (4) admits a solution satisfying* $y(-\infty) = -\sqrt{2},\ y(+\infty) = \sqrt{2}$.

The last conjecture, if true, would indicate that there is a biheteroclinic bifurcation which gives rise to infinitely many periodic solutions in the Kuramoto-Sivashinsky equation.

7 Final Comments

The method we present here

1. provides a relatively strong, stable under small perturbations and rigorous description of global dynamics of concrete differential equations.
2. may be used to prove the existence of stationary points, periodic trajectories, homoclinic and heteroclinic connections, chaotic dynamics and many other qualitative phenomena in nonlinear dynamics.
3. is relatively fast, due to the fact that it is based on computationally inexpensive topological conditions.
4. does not require high precision computations. The precision of 14 decimal digits offered by hardware is usually much more than necessary.
5. is still far from being fully optimized.
6. opens a broad area for future research on its effectiveness and range of practical applicability.

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